On the algebraic Bethe ansatz: Periodic boundary conditions

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Abstract

In this paper, the algebraic Bethe ansatz with periodic boundary conditions is used to investigate trigonometric vertex models associated with the fundamental representations of the non-exceptional Lie algebras. This formulation allow us to present explicit expressions for the eigenvectors and eigenvalues of the respective transfer matrices.

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1 Introduction

One of the main branches of theoretical and mathematical physics is the theory of exactly solvable models. The most successful approach to construct integrable two-dimensional lattice models of statistical mechanics and integrable $(1+1)$-dimensional quantum field theory is through the solution of the Yang-Baxter equation [1, 2]. Given a solution of this equation, depending on a spectral parameter λ, one can define the local Boltzmann weights of a commuting family of transfer matrix $T(\lambda)$ [3, 4] and the factorizable S-matrix in a two-dimensional quantum field theory [5].

The structure of the solutions of the Yang-Baxter equation based on simple Lie algebras is by now fairly well understood [6]. In particular, explicit expressions for the $\mathcal{R}$–matrices related to non-exceptional affine Lie algebras were exhibited in [7] and [8]. Since then, many other $\mathcal{R}$-matrices associated to higher dimensional representations of these algebras have also been determined [9].

A complete understanding of the vertex models living in a planar lattice include the exact diagonalization of the row-to-row transfer matrices which can provide informations about the on-shell physical properties such as free-energy thermodynamics and quasi-particle excitation behavior. The most efficient method for achieving this is the algebraic Bethe ansatz [10], though its coordinate version is usually more efficient for finding the energy spectrum of concrete models [11]. A long-standing open problem is the diagonalization of transfer matrices of vertex models associated with solutions of the Yang-Baxter equation.

One possible method of finding the eigenvalues of a given transfer matrix is the so-called analytical Bethe ansatz [12]. This technique relies on the unitarity, crossing and analyticity properties of the transfer matrix and, in some cases, an extra amount of phenomenological input is also required. This method has been applied to some of the models which we are going to consider in this paper, more precisely for the systems $\mathrm{B}_n^{(1)}$, $\mathrm{C}_n^{(1)}$, $\mathrm{D}_n^{(1)}$ [13, 14]. Unfortunately, the explicit construction of eigenvectors of the transfer matrix is beyond the scope of the analytical Bethe ansatz. The construction of exact eigenvectors is certainly an important step in the program of solving integrable models [15]

The importance of the algebraic Bethe ansatz does not rely on the calculation of the energy spectrum of a given model, but to also to supply information on the nature of the eigenfunctions. Thus is crucial in the investigation of off-shell properties such as correlators of physical operators [4] as well as for the calculation of form-factors [16].

A unified formulation of the quantum inverse scattering method for lattice vertex models associated to non-exceptional Lie algebras has been developed in the last years by Martins and collaborators [17, 18, 19]. In their works the Yang-Baxter algebra is recast in terms of novel commutation relations among creation, annihilation and diagonal fields. In particular, the solution of the twisted $\mathrm{D}_{n+1}^{(2)}$ vertex models is accommodated in their unification by the solution of a sixteen-vertex model [20].

In this work, we will describe a detailed account of this method which complements results on the literature [17], by the investigation of the trigonometric vertex models associated with the affine Lie algebras $\mathrm{B}_n^{(1)}$, $\mathrm{C}_n^{(1)}$, $\mathrm{D}_n^{(1)}$, $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{A}_{2n-1}^{(2)}$. The $\mathrm{D}_{n+1}^{(2)}$ vertex models are out of the scope of this paper.

The outline of the paper is as follows:

In section 2 we present the models through their $\mathcal{R}$ matrices. In section 3 the eigenvalue problem of the transfer matrix is formulated. In section 4 we perform a detailed construction of the intertwining relation in order to derive the fundamental commutation relations. In sections 5 and 6 the eigenvalue problem is executed in full detail for one and two-particle Bethe states, respectively, emphasizing the subtleties of each case and developing the language used in the text. In sections 7 the multi-particle cases are solved. The section 8 is reserved for our conclusions. In the appendix the $\mathrm{A}_n^{(1)}$ vertex models are considered for sake of completeness.

2 The vertex models

The search for integrable models through solutions of the Yang-Baxter equation has been performed by the quantum group approach in [22], where the problem is reduced to a linear one. Indeed, $\mathcal{R}$-matrices corresponding to vector representations of all non-exceptional affine Lie algebras were determined in this way in [8].

Quantum $\mathcal{R}$-matrices for the vertex models associated to the $\mathrm{B}_n^{(1)}$,$\mathrm{C}_n^{(1)}$, $\mathrm{D}_n^{(1)}$, $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{A}_{2n-1}^{(2)}$ affine Lie algebras as presented by Jimbo have the form [8]:

$$\begin{aligned}\mathcal{R}^{(l)} &= x_1^{(l)}\sum_{\alpha\neq\alpha'}^{N_l} E_{\alpha\alpha}\otimes E_{\alpha\alpha} + x_2^{(l)}\sum_{\alpha\neq\beta,\ \beta'}^{N_l} E_{\alpha\alpha}\otimes E_{\beta\beta} + x_3^{(l)}\sum_{\alpha<\beta,\ \alpha\neq\beta'}^{N_l} E_{\alpha\beta}\otimes E_{\beta\alpha} \\ &\quad + x_4^{(l)}\sum_{\alpha>\beta,\ \alpha\neq\beta'}^{N_l} E_{\alpha\beta}\otimes E_{\beta\alpha} + \sum_{\alpha,\beta}^{N_l} y_{\alpha\beta}^{(l)}\, E_{\alpha\beta}\otimes E_{\alpha'\beta'}\end{aligned} \tag{2.1}$$

where we have introduced a label l, $l = 0, 1, ..., n-1$ in order to work with the nesting structure presents in the nested Bethe Ansatz construction. For a given value of n, the label l identifies a particular vertex model among those models with $n-l \leq n$. We can name the label $l=0$ as the *ground* and the remained ones as the *layers* in the nest build. Here, E_{ij} denotes the elementary N_l by N_l matrices ($(E_{\alpha\beta})_{ab} = \delta_{\alpha a}\delta_{\beta b}$), where $N_l = 2(n-l)$ for $\mathrm{C}_{n-l}^{(1)}$, $\mathrm{D}_{n-l}^{(1)}$ and $\mathrm{A}_{2(n-l)-1}^{(2)}$ and $N_l = 2(n-l)+1$ for $\mathrm{B}_{n-l}^{(1)}$ and $\mathrm{A}_{2(n-l)}^{(2)}$.

The Boltzmann weights with functional dependence on the spectral parameter λ are given by

$$\begin{aligned} x_1^{(l)}(\lambda) &= (\mathrm{e}^\lambda - q^2)(\mathrm{e}^\lambda - \xi_l), \qquad x_2^{(l)}(\lambda) = q(\mathrm{e}^\lambda - 1)(\mathrm{e}^\lambda - \xi_l), \\ x_3^{(l)}(\lambda) &= -(q^2-1)(\mathrm{e}^\lambda - \xi_l), \qquad x_4^{(l)}(\lambda) = \mathrm{e}^\lambda x_3^{(l)}(\lambda)\end{aligned} \tag{2.2}$$

and

$$y_{\alpha\beta}^{(l)}(\lambda) = \begin{cases} (q^2\mathrm{e}^\lambda - \xi_l)(\mathrm{e}^\lambda - 1) & (\alpha=\beta,\ \alpha\neq\alpha') \\ q(\mathrm{e}^\lambda - \xi_l)(\mathrm{e}^\lambda - 1) + (\xi_l - 1)(q^2-1)\mathrm{e}^\lambda & (\alpha=\beta,\ \alpha=\alpha') \\ (q^2-1)\left(\varepsilon_\alpha\varepsilon_\beta\ \xi_l q^{\bar{\alpha}-\bar{\beta}}(\mathrm{e}^\lambda - 1) - \delta_{\alpha\beta'}(\mathrm{e}^\lambda - \xi_l)\right) & (\alpha<\beta) \\ (q^2-1)\mathrm{e}^\lambda\left(\varepsilon_\alpha\varepsilon_\beta\ q^{\bar{\alpha}-\bar{\beta}}(\mathrm{e}^\lambda - 1) - \delta_{\alpha\beta'}(\mathrm{e}^\lambda - \xi_l)\right) & (\alpha>\beta) \end{cases} \tag{2.3}$$

where $q = e^{-2\eta}$ denotes an arbitrary parameter and $\alpha' = N_l + 1 - \alpha$. The sign functions $\varepsilon_\alpha = 1$ $(1 \leq \alpha \leq n-l)$, $\epsilon_\alpha = -1$ $(n-l+1 \leq \alpha \leq 2(n-l))$ for $\mathrm{C}_{n-l}^{(1)}$ and $\varepsilon_\alpha = 1$ for the remaining cases.

Here ξ and $\bar{\alpha}$ are given respectively by

$$\xi_l = q^{2(n-l)-1},\ q^{2(n-l)+2},\ q^{2(n-l)-2},\ -q^{2(n-l)+1},\ -q^{2(n-l)} \tag{2.4}$$

for $\mathrm{B}_{n-l}^{(1)}$, $\mathrm{C}_{n-l}^{(1)}$, $\mathrm{D}_{n-l}^{(1)}$, $\mathrm{A}_{2(n-l)}^{(2)}$, $\mathrm{A}_{2(n-l)-1}^{(2)}$;

$$\bar{\alpha} = \begin{cases} \alpha - 1/2 & (1 \leq \alpha \leq n-l) \\ \alpha + 1/2 & (n-l+1 \leq \alpha \leq 2(n-l)) \end{cases} \tag{2.5}$$

for $\mathrm{C}_{n-l}^{(1)}$, and

$$\bar{\alpha} = \begin{cases} \alpha + 1/2 & (1 \leq \alpha < \frac{N_l+1}{2}) \\ \alpha & (\ \alpha = \frac{N_l+1}{2}\ ) \\ \alpha - 1/2 & (\frac{N_l+1}{2} < \alpha \leq N_l) \end{cases} \tag{2.6}$$

in the remaining cases.

These $\mathcal{R}$-matrices are regular satisfying PT-symmetry and unitarity:

$$\mathcal{R}^{(l)}(0) = x_1^{(l)}(0)\mathcal{P}^{(l)}, \qquad \mathcal{R}_{21}^{(l)}(\lambda) = \mathcal{P}_{12}^{(l)}\mathcal{R}_{12}^{(l)}(\lambda)\mathcal{P}_{12}^{(l)}, \qquad \mathcal{R}_{12}^{(l)}(\lambda)\mathcal{R}_{21}^{(l)}(-\lambda) = x_1^{(l)}(\lambda)x_1^{(l)}(-\lambda), \tag{2.7}$$

where $\mathcal{P}$ is the permutation matrix: $\mathcal{P}^{(l)}\, |\alpha\rangle \otimes |\beta\rangle = |\beta\rangle \otimes |\alpha\rangle$.

3 The eigenvalue problem

In the context of two-dimensional classical statistical systems, each Yang-Baxter solution $\mathcal{R}(\lambda)$ is interpreted as local Boltzmann weights of an integrable vertex model on a square lattice of size $L \times L$. A physical state on this lattice is defined by the assignment of a *state variable* to each lattice edge. If one takes the horizontal direction as space and the vertical one as time, the transfer matrix $\tau_L(\lambda)$ plays the role of a discrete evolution operator acting on the Hilbert space $\mathcal{H}^{(L)}$ spanned by the *row states* which are defined by the set of vertical link variables on the same row. Thus, the transfer matrix elements can be understood as the transition probability of the one row state to project on the consecutive one after a unit of time.

The main problem is the diagonalization of the $\tau_L(\lambda)$ matrix for these lattice systems. To do this we request the algebraic Bethe ansatz where the row-to-row transfer matrix can be constructed from a local vertex operator $\mathcal{L}_{ai}(\lambda)$, the Lax operator, which is viewed as a matrix on the N-dimensional auxiliary space V_a, corresponding in the vertex model to the space of states of the horizontal degrees of freedom. Its matrix elements are operators on the L-product Hilbert space $\mathcal{H}^{(L)} = V_i^{\otimes L}$, where V_i corresponds to the space of vertical degrees of freedom and i denotes the sites of the one-dimensional lattice. An ordered product of Lax operators defines the N^{2L} by N^{2L} monodromy matrix

$$T(\lambda) = \mathcal{L}_{aL}(\lambda)\mathcal{L}_{aL-1}(\lambda)\cdots\mathcal{L}_{a1}(\lambda). \tag{3.1}$$

which can be written as an N by N matrix with entries

$$T_{ij}(\lambda) = \sum_{k_1,\ldots,k_{L-1}=1}^{L} \mathcal{L}_{ik_1}^{(L)}(\lambda) \otimes \mathcal{L}_{k_1k_2}^{(L-1)}(\lambda) \otimes \cdots \otimes \mathcal{L}_{k_{L-1}j}^{(1)}(\lambda) \tag{3.2}$$

where $\mathcal{L}_{ij}^{(n)}(\lambda)$ are N by N matrices acting on the quantum space V_n.

The transfer matrix of the vertex model with periodic boundary conditions can be written as a trace of the monodromy matrix on the auxiliary space V_a

$$\tau_L(\lambda) = \mathrm{Tr}_a[T(\lambda)] = \sum_{i=1}^{N} T_{ii}(\lambda) \tag{3.3}$$

and the eigenvalue problem is defined by

$$\tau_L(\lambda)\Psi = \Lambda(\lambda)\Psi \tag{3.4}$$

where the eigenfunction Ψ is obtained from the action of the non-diagonal matrix elements of $T(\lambda)$ on a reference state.

Here one uses the fact that the Yang-Baxter equation

$$\mathcal{R}_{12}(\lambda-\mu)\mathcal{R}_{13}(\lambda)\mathcal{R}_{23}(\mu) = \mathcal{R}_{23}(\mu)\mathcal{R}_{13}(\lambda)\mathcal{R}_{12}(\lambda-\mu). \tag{3.5}$$

can be recast in the form of commutation relations for the matrix elements of the monodromy matrix which play the role of creation and annihilation operators. The commutation relations are derived from the global intertwining relation

$$S(\lambda-\mu)T(\lambda)\otimes T(\mu)=T(\mu)\otimes T(\lambda)S(\lambda-\mu) \tag{3.6}$$

where we have used that the intertwining matrix $S(\lambda-\mu)$ is defined on the tensor product $V_a\otimes V_a$ and satisfy the relation $S(\lambda-\mu)=\mathcal{P}\mathcal{R}(\lambda-\mu)$ when the auxiliary space V_a and the quantum space V_i are equivalent and the Lax operator identified with the $\mathcal{R}$ matrix, i.e., $\mathcal{L}(\lambda)\doteq\mathcal{R}(\lambda)$. The indices in the matrix $\mathcal{R}$ denote the spaces where its action is not trivial.

In this paper a sufficiently general recipe is supplied to derive the fundamental commutation relations among the monodromy elements for the trigonometric vertex models associated with the $\mathrm{B}_n^{(1)}$, $\mathrm{C}_n^{(1)}$, $\mathrm{D}_n^{(1)}$, $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{A}_{2n-1}^{(2)}$ affine Lie algebras.

For the vertex models (2.1) the monodromy matrix (3.3) can be written as an N_l by N_l matrix

$$T^{(l)}=\begin{pmatrix} A_1^{(l)} & B_1^{(l)} & B_2^{(l)} & \cdots & B_{N_l-2}^{(l)} & B_{N_l-1}^{(l)} \\ C_1^{(l)} & D_{1,1}^{(l)} & D_{1,2}^{(l)} & \cdots & D_{1,N_l-2}^{(l)} & B_{N_l}^{(l)} \\ C_2^{(l)} & D_{2,1}^{(l)} & D_{2,2}^{(l)} & \cdots & D_{2,N_l-2}^{(l)} & B_{N_l+1}^{(l)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{N_l-2}^{(l)} & D_{N_l-2,1}^{(l)} & D_{N_l-2,2}^{(l)} & \cdots & D_{N_l-2,N_l-2}^{(l)} & B_{2N_l-3}^{(l)} \\ C_{N_l-1}^{(l)} & C_{N_l}^{(l)} & C_{N_l+1}^{(l)} & \cdots & C_{2N_l-3}^{(l)} & A_3^{(l)} \end{pmatrix}. \tag{3.7}$$

The usual reference state

$$|0_L\rangle^{(l)}=\prod_{i=1}^{L}\otimes|0\rangle_i\,,\qquad |0\rangle_i=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}_{N_l} \tag{3.8}$$

where N_l is the length of the vectors $|0\rangle_i$, is the highest vector of $T^{(l)}(\lambda)$

$$\begin{aligned} A_1^{(l)}(\lambda)\,|0_L\rangle^{(l)} &= X_1^{(l)}(\lambda)\,|0_L\rangle^{(l)}\,, \quad A_3^{(l)}(\lambda)\,|0_L\rangle^{(l)}=X_3^{(l)}(\lambda)\,|0_L\rangle^{(l)} \\ D_{\alpha\alpha}^{(l)}(\lambda)\,|0_L\rangle^{(l)} &= X_2^{(l)}(\lambda)\,|0_L\rangle^{(l)}\,, \quad D_{\alpha\beta}^{(l)}(\lambda)\,|0_L\rangle^{(l)}=0, \\ B_\alpha^{(l)}(\lambda)\,|0_L\rangle^{(l)} &\neq \{0,\ |0_L\rangle^{(l)}\}, \qquad C_\alpha^{(l)}(\lambda)\,|0_L\rangle^{(l)}=0, \\ \alpha &\neq \beta=1,2,\ldots,N_l-2 \end{aligned} \tag{3.9}$$

where

$$X_1^{(l)}(\lambda)=[x_1^{(l)}(\lambda)]^L,\qquad X_2^{(l)}(\lambda)=[x_2^{(l)}(\lambda)]^L,\qquad X_3^{(l)}(\lambda)=[y_{N_lN_l}^{(l)}(\lambda)]^L \tag{3.10}$$

This triangular property suggests to write $T^{(l)}(\lambda)$ as a 3 by 3 matrix:

$$T^{(l)}(\lambda)=\begin{pmatrix} A_1^{(l)}(\lambda) & \mathcal{B}^{(l)}(\lambda) & B_{N_l-1}(\lambda) \\ \mathcal{C}^{(l)}(\lambda) & \mathcal{D}^{(l)}(\lambda) & \mathcal{B}^{*(l)}(\lambda) \\ C_{N_l-1}(\lambda) & \mathcal{C}^{*(l)}(\lambda) & A_3^{(l)}(\lambda) \end{pmatrix} \tag{3.11}$$

where one can identify four scalars

$$A_1^{(l)}(\lambda),\qquad B_{N_l-1}^{(l)}(\lambda),\qquad C_{N_l-1}^{(l)}(\lambda),\qquad A_3^{(l)}(\lambda), \tag{3.12}$$

as well as, four (N_l-2)-dimensional vectors

$$\mathcal{B}^{(l)}(\lambda)=\left(B_1^{(l)}(\lambda),B_2^{(l)}(\lambda),\cdots,B_{N_l-2}^{(l)}(\lambda)\right),\qquad \mathcal{B}^{*(l)}(\lambda)=\begin{pmatrix} B_{N_l}^{(l)}(\lambda)\\ B_{N_l+1}^{(l)}(\lambda)\\ \vdots\\ B_{2N_l-3}^{(l)}(\lambda)\end{pmatrix}, \tag{3.13}$$

$$\mathcal{C}^{(l)}(\lambda)=\begin{pmatrix} C_1^{(l)}(\lambda)\\ C_2^{(l)}(\lambda)\\ \vdots\\ C_{N_l-2}^{(l)}(\lambda)\end{pmatrix},\qquad \mathcal{C}^{*(l)}(\lambda)=\left(C_{N_l}^{(l)}(\lambda),C_{N_l+1}^{(l)}(\lambda),\cdots,C_{2N_l-3}^{(l)}(\lambda)\right), \tag{3.14}$$

besides an $(N_l-2\ )$ by (N_l-2) matrix denoted by

$$\mathcal{D}^{(l)}(\lambda)=\begin{pmatrix} D_{11}^{(l)}(\lambda) & D_{12}^{(l)}(\lambda) & \cdots & D_{1,N_l-2}^{(l)}(\lambda)\\ D_{21}^{(l)}(\lambda) & D_{22}^{(l)}(\lambda) & \cdots & D_{2,N_l-2}^{(l)}(\lambda)\\ \vdots & \vdots & \ddots & \vdots\\ D_{N_l-2,1}^{(l)}(\lambda) & D_{N_l-2,2}^{(l)}(\lambda) & \cdots & D_{N_l-2,N_l-2}^{(l)}(\lambda)\end{pmatrix}. \tag{3.15}$$

The problem of diagonalization of the transfer matrix becomes

$$\tau_L^{(l)}(\lambda)\Psi_m^{(l)}=\left[A_1^{(l)}(\lambda)+\sum_{\alpha=1}^{N_l-2}D_{\alpha\alpha}^{(l)}(\lambda)+A_3^{(l)}(\lambda)\right]\Psi_m^{(l)}=\Lambda_L^{(l)}(\lambda|\{\lambda_i\})\Psi_m^{(l)}. \tag{3.16}$$

where $\Psi_m^{(l)}=\Psi_m^{(l)}(\lambda_1,...,\lambda_m)$ is a scalar function named the m-particle state for the eigenvalue problem. In particular, the eigenvalue of the reference state (3.8) is

$$\Lambda_L^{(l)}(\lambda|0)=[x_1^{(l)}(\lambda)]^L+(N_l-2)[x_2^{(l)}(\lambda)]^L+[y_{N_lN_l}^{(l)}(\lambda)]^L. \tag{3.17}$$

In order to construct other eigenvalues, one has to find the commutation relations among the elements of $T^{(l)}(\lambda)$.

4 Fundamental Commutation Relations

In contrast to what happens to the six-vertex model and its multi-states generalizations it is a rather complicated task to find the commutation relations among the matrix elements of the monodromy matrix in a general case. However, as we are going to see, this construction can be simplified because all informations about the commutation relations are already encoded in the simplest cases.

For $N_l \geq 3$ the corresponding N_l by N_l intertwining $S^{(l)}$ matrix can also be suitably written as a 9 by 9 matrix in the form

$$[S^{(l)}] = \begin{pmatrix} x_1^{(l)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_4^{(l)} & 0 & x_2^{(l)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_{N_l 1}^{(l)} & 0 & \hat{Y}_{N_l 2}^{(l)} & 0 & y_{N_l N_l}^{(l)} & 0 & 0 \\ 0 & x_2^{(l)} & 0 & x_3^{(l)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{Y}_{21}^{(l)} & 0 & \hat{Y}^{(l)} & 0 & \hat{Y}_{2N_l}^{(l)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_4^{(l)} & 0 & x_2^{(l)} & 0 \\ 0 & 0 & y_{11}^{(l)} & 0 & \hat{Y}_{12}^{(l)} & 0 & y_{1N_l}^{(l)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2^{(l)} & 0 & x_3^{(l)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1^{(l)} \end{pmatrix} \tag{4.1}$$

where one can identify four N_{l+1}^2-dimensional vectors

$$\hat{Y}_{N_l,2}^{(l)}(\lambda) = \sum_{i=1}^{N_l-2} y_{N_l,i+1}^{(l)}(\lambda)\ (E_i \otimes E_{i'})^t, \quad \hat{Y}_{2N_l}^{(l)}(\lambda) = \sum_{i=1}^{N_l-2} y_{i+1,N_l}^{(l)}(\lambda)\ E_{i'} \otimes E_i \tag{4.2}$$

$$\hat{Y}_{12}^{(l)}(\lambda) = \sum_{i=1}^{N_l-2} y_{1,i+1}^{(l)}(\lambda)\ (E_i \otimes E_{i'})^t, \qquad \hat{Y}_{21}^{(l)}(\lambda) = \sum_{i=1}^{N_l-2} y_{i+1,1}^{(l)}(\lambda)\ E_{i'} \otimes E_i \tag{4.3}$$

and a N_{l+1}^2 by N_{l+1}^2 matrix $\hat{Y}^{(l)} = \mathcal{P}^{(l+1)} Y^{(l)}$, with $Y^{(l)}$ obtained from the $\mathcal{R}^{(l)}$ matrix (2.1) by the reduction

$$\begin{aligned} Y^{(l)} \quad &= \quad x_1^{(l)} \sum_{i \neq i'}^{N_l-2} E_{ii} \otimes E_{ii} + x_2^{(l)} \sum_{i \neq j, j'}^{N_l-2} E_{ii} \otimes E_{jj} + x_3^{(l)} \sum_{i<j, i \neq j'}^{N_l-2} E_{ij} \otimes E_{ji} \\ &\quad + x_4^{(l)} \sum_{i>j, i \neq j'}^{N_l-2} E_{ij} \otimes E_{ji} + \sum_{i,j=1}^{N_l-2} y_{i+1 j+1}^{(l)}\ E_{i,j} \otimes E_{i',j'} \end{aligned} \tag{4.4}$$

The remaining entries of $[S^{(l)}]$ are five scalars $\{x_1^{(l)}, y_{11}^{(l)}, y_{1N_l}^{(l)}, y_{N_l 1}^{(l)}, y_{N_l N_l}^{(l)}\}$ and three N_{l+1} by N_{l+1} identity matrices $\{x_2^{(l)}, x_3^{(l)}, x_4^{(l)}\}$. The E_i in the definition relations of the $\hat{Y}_{i,j}^{(l)}$ are column vectors of length N_{l+1} with only unitary element at ith position and t means transposition.

Here we note that although these vectors and matrices act on the *layer* $l+1$ ($N_{l+1} = N_l - 2$), their Boltzmann weights are written in term of the l-*th* layer Boltzmann weights. Therefore, our label l in a particular Boltzmann weight is indicating the model that it belongs.

Now, using (4.1) and (3.11) we can write the fundamental relation in the form

$$[S^{(l)}](\lambda - \mu) T^{(l)}(\lambda) \otimes T^{(l)}(\mu) = T^{(l)}(\mu) \otimes T^{(l)}(\lambda) [S^{(l)}](\lambda - \mu) \tag{4.5}$$

in order to get 81 equations for the commutation relations among the entries of the monodromy matrix.

Before we look at these equations we would like to show what happens in the simplest cases. For $N_l = 3$, all entries of $[S^{(l)}]$ are reduced to the scalar status and we have

$$[S] = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_4 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_{31} & 0 & y_{32} & 0 & y_{33} & 0 & 0 \\ 0 & x_2 & 0 & x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_{21} & 0 & y_{22} & 0 & y_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_4 & 0 & x_2 & 0 \\ 0 & 0 & y_{11} & 0 & y_{12} & 0 & y_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2 & 0 & x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \end{pmatrix} \tag{4.6}$$

This is the intertwining S matrix for the $\mathrm{B}_1^{(1)}$ and $\mathrm{A}_2^{(2)}$ vertex models whose $\mathcal{R}$-matrices are given by (2.1) with $N = 3$. Moreover, the matrix elements of (3.11) are scalars in the auxiliary space

$$T = \begin{pmatrix} A_1 & B_1 & B_2 \\ C_1 & D_{11} & B_3 \\ C_2 & C_3 & A_3 \end{pmatrix} \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.5) we get the commutation rules for the entries of (4.7) proposed previously by Tarasov in the context of the Izergin-Korepin vertex model [23].

It is also worth note that the case $N_l = 2$ can be added to our discussion. In this case all entries with tensor status are removed from $[S^{(l)}]$. The result is

$$[S] = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & y_{21} & y_{22} & 0 \\ 0 & y_{11} & y_{12} & 0 \\ 0 & 0 & 0 & x_1 \end{pmatrix} \tag{4.8}$$

which is the intertwining S matrix for the $\mathrm{C}_1^{(1)}$,$\mathrm{D}_1^{(1)}$ and $\mathrm{A}_1^{(2)}$ vertex models whose $\mathcal{R}$-matrices are also given by (2.1). Consequently, their monodromy matrices preserve only the scalar entries (3.12):

$$T = \begin{pmatrix} A_1 & B_1 \\ C_1 & A_3 \end{pmatrix} \tag{4.9}$$

Of course this is not a simple limit from the general case because their Bethe states are different by construction. However, we can derive the commutation relations for the entries of (4.9) by vanishing all entries which are vector or matrix in the general commutation relations which we are going to derive. Note also that $\mathrm{C}_1^{(1)}$ and $\mathrm{A}_1^{(2)}$ are six-vertex models while $\mathrm{D}_1^{(1)}$ is neither a six-vertex model nor a regular model as we can see from they $\mathcal{R}$ matrices.

In order to derive these so important commutation relations we shall proceed in the following way: we denote by $E[i,j] = 0$ the (i,j) component of the matrix equation (4.5) and collecting them in 25 blocks $B[i,j]$, $(i = 1, ..., 5,\ j = i, ..., 10 - i)$, defined by

$$B[i,j] = \begin{cases} E_{ij} = E[i,j], & e_{ij} = E[j,i], \\ \bar{E}_{ij} = E[10-i, 10-j, & \bar{e}_{ij} = E[10-j, 10-i] \end{cases} \tag{4.10}$$

For a given block $B[i,j]$, the equation $\bar{e}_{ij}$ can be read from the equation E_{ij} (and the equation $\bar{E}_{ij}$ from

the equation e_{ij}) by the interchanging

$$
\begin{aligned}
A_1^{(l)}(\lambda) &\leftrightarrow A_3^{(l)}(\mu), \quad B_{N_l-1}(\lambda) \leftrightarrow B_{N_l-1}(\mu), \quad \mathcal{B}^{(l)}(\lambda) \leftrightarrow \mathcal{B}^{*(l)}(\mu), \\
\mathcal{C}^{(l)}(\lambda) &\leftrightarrow \mathcal{C}^{*(l)}(\mu), \quad C_{N_l-1}(\lambda) \leftrightarrow C_{N_l-1}(\mu), \quad \mathcal{D}^{(l)}(\lambda) \leftrightarrow \mathcal{D}^{(l)}(\mu), \\
x_4^{(l)} &\leftrightarrow x_3^{(l)}, \quad y_{1,N_l}^{(l)} \leftrightarrow y_{N_l,1}^{(l)} \quad \hat{Y}_{21}^{(l)} \leftrightarrow \hat{Y}_{12}^{(l)}, \quad \hat{Y}_{N_l,2}^{(l)} \leftrightarrow \hat{Y}_{2,N_l}^{(l)}
\end{aligned}
\tag{4.11}
$$

Here we note that the Boltzmann weights (2.3) satisfy the relation $y_{\alpha,\beta}^{(l)}(\lambda) = y_{\beta',\alpha'}^{(l)}(\lambda)$. Taking into account these identifications, the computation to find commutation relations is considerably simplified.

For instance, in the block $B[1,4]$ we can solve the equations $E_{14}=0$ and $\bar{e}_{14}=0$ in order to find

$$
A_1^{(l)}(\lambda)\mathcal{B}^{(l)}(\mu) = z^{(l)}(\mu-\lambda)\mathcal{B}^{(l)}(\mu)A_1^{(l)}(\lambda) - \frac{x_3^{(l)}(\mu-\lambda)}{x_2^{(l)}(\mu-\lambda)}\mathcal{B}^{(l)}(\lambda)A_1^{(l)}(\mu) \tag{4.12}
$$

and

$$
A_3^{(l)}(\lambda)\mathcal{B}^{*(l)}(\mu) = z^{(l)}(\lambda-\mu)\mathcal{B}^{*(l)}(\mu)A_3^{(l)}(\lambda) - \frac{x_4^{(l)}(\lambda-\mu)}{x_2^{(l)}(\lambda-\mu)}\mathcal{B}^{*(l)}(\lambda)A_3^{(l)}(\mu). \tag{4.13}
$$

where we have introduced the function

$$
z^{(l)}(\lambda) = \frac{x_1^{(l)}(\lambda)}{x_2^{(l)}(\lambda)} \tag{4.14}
$$

since it will appears many times in the text.

Let us consider here two further blocks: the equations in the block $B[2,7]$ yield the following intertwining relations

$$
\begin{aligned}
A_1^{(l)}(\lambda)\mathcal{B}^{*(l)}(\mu) \;=\; & \frac{x_2^{(l)}(\mu-\lambda)}{y_{N_lN_l}^{(l)}(\mu-\lambda)}\mathcal{B}^{*(l)}(\mu)A_1^{(l)}(\lambda) - \mathcal{B}^{(l)}(\lambda)\otimes\mathcal{D}^{(l)}(\mu)\frac{\hat{Y}_{2N_l}^{(l)}(\mu-\lambda)}{y_{N_lN_l}^{(l)}(\mu-\lambda)} \\
& + \frac{x_4^{(l)}(\mu-\lambda)}{y_{N_lN_l}^{(l)}(\mu-\lambda)}B_{N_l-1}(\mu)\mathcal{C}^{(l)}(\lambda) - \frac{y_{1N_l}^{(l)}(\mu-\lambda)}{y_{N_lN_l}^{(l)}(\mu-\lambda)}B_{N_l-1}(\lambda)\mathcal{C}^{(l)}(\mu)
\end{aligned}
\tag{4.15}
$$

and

$$
\begin{aligned}
A_3^{(l)}(\lambda)\mathcal{B}^{(l)}(\mu) \;=\; & \frac{x_2^{(l)}(\lambda-\mu)}{y_{N_lN_l}^{(l)}(\lambda-\mu)}\mathcal{B}^{(l)}(\mu)A_3^{(l)}(\lambda) - \frac{\hat{Y}_{N_l2}^{(l)}(\lambda-\mu)}{y_{N_lN_l}^{(l)}(\lambda-\mu)}\mathcal{B}^{*(l)}(\lambda)\otimes\mathcal{D}^{(l)}(\mu) \\
& + \frac{x_3^{(l)}(\lambda-\mu)}{y_{N_lN_l}^{(l)}(\lambda-\mu)}B_{N_l-1}(\mu)\mathcal{C}^{*(l)}(\lambda) - \frac{y_{N_l1}^{(l)}(\lambda-\mu)}{y_{N_lN_l}^{(l)}(\lambda-\mu)}B_{N_l-1}(\lambda)\mathcal{C}^{*(l)}(\mu)
\end{aligned}
\tag{4.16}
$$

where the tensor structure coming from (4.5) with scalars, vectors and matrices defined previously. It is also worth note the correspondence between (4.15) and (4.16) via the exchange property (4.11) and the matrices product order.

A very important information is given by the commutation relations derived from the block $B[2,5]$

$$\begin{aligned}
\mathcal{D}^{(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\mu) &= \mathcal{B}^{(l)}(\mu)\otimes\mathcal{D}^{(l)}(\lambda)\frac{s^{(l)}(\lambda-\mu)}{x_2^{(l)}(\lambda-\mu)} - B_{N_l-1}(\mu)\mathcal{C}^{(l)}(\lambda)\frac{\hat{Y}^{(l)}_{N_l2}(\mu-\lambda)}{y^{(l)}_{N_lN_l}(\mu-\lambda)}\frac{s^{(l)}(\lambda-\mu)}{x_2^{(l)}(\lambda-\mu)} \\
& -\frac{x_4^{(l)}(\lambda-\mu)}{x_2^{(l)}(\lambda-\mu)}\mathcal{B}^{(l)}(\lambda)\otimes\mathcal{D}^{(l)}(\mu) + \frac{x_4^{(l)}(\lambda-\mu)}{x_2^{(l)}(\lambda-\mu)}B_{N_l-1}(\lambda)\mathcal{C}^{(l)}(\mu)\frac{\hat{Y}^{(l)}_{N_l2}(\lambda-\mu)}{y^{(l)}_{N_lN_l}(\lambda-\mu)} \\
& +\mathcal{B}^{*(l)}(\lambda)A_1^{(l)}(\mu)\frac{\hat{Y}^{(l)}_{N_l2}(\lambda-\mu)}{y^{(l)}_{N_lN_l}(\lambda-\mu)} \qquad (4.17)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}^{(l)}(\lambda)\otimes\mathcal{B}^{*(l)}(\mu) &= \frac{s^{(l)}(\mu-\lambda)}{x_2^{(l)}(\mu-\lambda)}\mathcal{B}^{*(l)}(\mu)\otimes\mathcal{D}^{(l)}(\lambda) - \frac{s^{(l)}(\mu-\lambda)}{x_2^{(l)}(\mu-\lambda)}\frac{\hat{Y}^{(l)}_{2N_l}(\lambda-\mu)}{y^{(l)}_{N_lN_l}(\lambda-\mu)}B_{N_l-1}(\mu)\mathcal{C}^{*(l)}(\lambda) \\
& -\frac{x_3^{(l)}(\mu-\lambda)}{x_2^{(l)}(\mu-\lambda)}\mathcal{B}^{*(l)}(\lambda)\otimes\mathcal{D}^{(l)}(\mu) + \frac{x_3^{(l)}(\mu-\lambda)}{x_2^{(l)}(\mu-\lambda)}\frac{\hat{Y}^{(l)}_{2N_l}(\mu-\lambda)}{y^{(l)}_{N_lN_l}(\mu-\lambda)}B_{N_l-1}(\lambda)\mathcal{C}^{*(l)}(\mu) \\
& +\frac{\hat{Y}^{(l)}_{2N_l}(\mu-\lambda)}{y^{(l)}_{N_lN_l}(\mu-\lambda)}\mathcal{B}^{(l)}(\lambda)A_3^{(l)}(\mu) \qquad (4.18)
\end{aligned}$$

where we have defined a N_{l+1}^2 by N_{l+1}^2 matrix

$$\begin{aligned}
s^{(l)}(\lambda) &= \hat{Y}^{(l)}(\lambda) - \frac{1}{y^{(l)}_{N_lN_l}(\lambda)}\hat{Y}^{(l)}_{N_l2}(\lambda)\otimes\hat{Y}^{(l)}_{2N_l}(\lambda) \\
&= \hat{Y}^{(l)}(\lambda) - \frac{1}{y^{(l)}_{N_lN_l}(\lambda)}\hat{Y}^{(l)}_{2N_l}(\lambda)\otimes\hat{Y}^{(l)}_{N_l2}(\lambda) \qquad (4.19)
\end{aligned}$$

which satisfies the permuted version of the Yang-Baxter equation

$$s^{(l)}_{12}(\lambda)s^{(l)}_{23}(\lambda+\mu)s^{(l)}_{12}(\lambda) = s^{(l)}_{23}(\mu)s^{(l)}_{12}(\lambda+\mu)s^{(l)}_{23}(\mu) \qquad (4.20)$$

In the definition (4.19) the entries of $s^{(l)}(\lambda)$ were written in terms of the Boltzmann weight of the vertex model with label l. However, due to (4.20), its label must be $l+1$. To write the matrix $s(\lambda)$ with its Boltzmann weights labeled correctly we can use the identities

$$\frac{s^{(l)}(\lambda)}{x_a^{(l)}(\lambda)} = \frac{S^{(l+1)}(\lambda)}{x_a^{(l+1)}(\lambda)}, \qquad a=1,2. \qquad (4.21)$$

where $S^{(l+1)} = \mathcal{P}^{(l+1)}\mathcal{R}^{(l+1)}$ and $\mathcal{R}^{(l+1)}$ is given by (2.1) replacing l by $l+1$. These relations give the emphasis of the meaning of the label l. Moreover, in (4.17) and (4.18) we have used the following matrix identities

$$\begin{aligned}
\frac{\hat{Y}^{(l)}_{N_l2}(\mu-\lambda)}{y^{(l)}_{N_lN_l}(\mu-\lambda)}s^{(l)}(\lambda-\mu) &= -\hat{Y}^{(l)}_{12}(\lambda-\mu) + \frac{y^{(l)}_{1N_l}(\lambda-\mu)}{y^{(l)}_{N_lN_l}(\lambda-\mu)}\hat{Y}^{(l)}_{N_l2}(\lambda-\mu) \\
s^{(l)}(\lambda-\mu)\frac{\hat{Y}^{(l)}_{2N_l}(\mu-\lambda)}{y^{(l)}_{N_lN_l}(\mu-\lambda)} &= -\hat{Y}^{(l)}_{21}(\lambda-\mu) + \frac{y^{(l)}_{N_l1}(\lambda-\mu)}{y^{(l)}_{N_lN_l}(\lambda-\mu)}\hat{Y}^{(l)}_{2N_l}(\lambda-\mu) \qquad (4.22)
\end{aligned}$$

and the scalar relation

$$\frac{x_4^{(l)}(\lambda)}{x_2^{(l)}(\lambda)}+\frac{x_3^{(l)}(-\lambda)}{x_2^{(l)}(-\lambda)}=0 \tag{4.23}$$

Many other commutation relations will be used in this paper. In the right time we will derive them recalling the block $B[i,j]$ once more.

5 The one-particle Bethe state

In the quantum inverse scattering method the eigenstates of the transfer matrix are constructing by the action of the creators operators on the reference state. Such procedure results in excitations with multi-particle structure, characterized by a set of rapidities $\{\lambda_i\}$ which are determined by solving the Bethe equations. In the general case the vector $\mathcal{B}^{(l)}(\mu)$ has N_l-2 components and is used to define the one-particle state which is a scalar function obtained by the linear combination

$$\Psi_1^{(l)}(\lambda_1)=\mathcal{B}^{(l)}(\lambda_1)\mathcal{F}_1^{(l)}\left|0_L^{(l)}\right\rangle \tag{5.1}$$

where $\mathcal{F}_1^{(l)}$ is a vector with N_{l+1} components $f^{(l)\alpha}$.

The action of the transfer matrix $\tau_L^{(l)}(\lambda)$ on this state is determined by (3.9) and the intertwining relations (4.5). The components of (4.5) needed for the construction of the nested Bethe ansatz of the one-particle state are the commutation relations (4.12), (4.16) and (4.17). In particular, the matrix relation (4.17) must be written in terms of its components in order to get the commutation relation for the scalar $\sum_\alpha D_{\alpha\alpha}^{(l)}(\lambda)$:

$$\begin{aligned}
\sum_{\alpha=1}^{N_l-2} D_{\alpha\alpha}^{(l)}(\lambda)\mathcal{B}^{(l)}(\mu) \quad = \quad & \mathcal{B}^{(l)}(\mu)\mathrm{Tr}_a\left[\frac{\mathcal{L}_{a1}^{(l+1)}(\lambda-\mu)}{x_2^{(l+1)}(\lambda-\mu)}\mathcal{D}^{(l)}(\lambda)\right]-\frac{x_4^{(l)}(\lambda-\mu)}{x_2^{(l)}(\lambda-\mu)}\mathcal{B}^{(l)}(\lambda)\mathcal{D}^{(l)}(\mu)\\
&+\frac{\hat{Y}_{N_l2}^{(l)}(\lambda-\mu)}{y_{N_lN_l}^{(l)}(\lambda-\mu)}[B^{*(l)}(\lambda)\otimes\mathbf{1}^{(l)}]A_1^{(l)}(\mu)\\
&-B_{N_l-1}(\mu)\frac{\hat{Y}_{N_l2}^{(l)}(\mu-\lambda)}{y_{N_lN_l}^{(l)}(\mu-\lambda)}\frac{S^{(l+1)}(\lambda-\mu)}{x_2^{(l+1)}(\lambda-\mu)}[\mathcal{C}^{(l)}(\lambda)\otimes\mathbf{1}^{(l)}]\\
&+B_{N_l-1}(\lambda)\frac{x_4^{(l)}(\lambda-\mu)}{x_2^{(l)}(\lambda-\mu)}\frac{\hat{Y}_{N_l2}^{(l)}(\lambda-\mu)}{y_{N_lN_l}^{(l)}(\lambda-\mu)}[\mathcal{C}^{(l)}(\mu)\otimes\mathbf{1}^{(l)}]
\end{aligned} \tag{5.2}$$

where $1^{(l)}$ is the N_{l+1} by N_{l+1} matrix identity and Tr_a is the trace in the auxiliary space. In (5.2), we also have used (4.21) to write the matrix $s^{(l)}$ as $S^{(l+1)}$ and identified its permuted matrix $\mathcal{R}^{(l+1)}$ with the Lax operator $\mathcal{L}^{(l+1)}$.

In the nested Bethe ansatz procedure we always begin at the ground level $l=0$ and from now we shall omitte such label in the expressions of the first eigenvalue problem.

The eigenvalue problem is accomplished by the action of $\tau_L(\lambda)$ on $\Psi_1(\lambda_1)$

$$\begin{aligned}
\tau_L(\lambda)\Psi_1(\lambda_1) \quad &= \quad \left(A_1(\lambda)+\sum_{\alpha=1}^{N-2}D_{\alpha\alpha}(\lambda)+A_3(\lambda)\right)\mathcal{B}(\lambda_1)\mathcal{F}_1\left|0_L\right\rangle\\
&\doteq \quad \Lambda_L(\lambda|\{\lambda_1\})\Psi_1(\lambda_1)
\end{aligned} \tag{5.3}$$

Taking into account the commutation relations (4.12), (4.16) , (5.2) and (3.9) we are able to turn the operators $A_1(\lambda)$, $D_{\alpha\alpha}(\lambda)$ and $A_3(\lambda)$ over the creation operator $\mathcal{B}(\lambda_1)$ and as result we have the following

expression for the first eigenvalue problem:

$$
\begin{aligned}
\tau_L(\lambda)\Psi_1(\lambda_1) &= X_1(\lambda)z(\lambda_1-\lambda)\Psi_1(\lambda_1)+X_3(\lambda)\frac{x_2(\lambda-\lambda_1)}{y_{NN}(\lambda-\lambda_1)}\Psi_1(\lambda_1) \\
&+\mathcal{B}(\lambda_1)\mathrm{Tr}_a\left[\frac{\mathcal{L}^{(1)}_{a1}(\lambda-\lambda_1)}{x^{(1)}_2(\lambda-\lambda_1)}\mathcal{D}(\lambda)\right]\mathcal{F}_1\left|0_L\right\rangle \\
&-\mathcal{B}(\lambda)[X_1(\lambda_1)\frac{x_3(\lambda_1-\lambda)}{x_2(\lambda_1-\lambda)}+X_2(\lambda_1)\frac{x_4(\lambda-\lambda_1)}{x_2(\lambda-\lambda_1)}]\mathcal{F}_1\left|0_L\right\rangle \\
&-\frac{\hat{Y}_{N2}(\lambda-\lambda_1)}{y_{NN}(\lambda-\lambda_1)}\mathcal{B}^*(\lambda)\otimes\mathbf{1}[X_2(\lambda_1)-X_1(\lambda_1)]\mathcal{F}_1\left|0_L\right\rangle
\end{aligned} \tag{5.4}
$$

Of course, the terms proportional to $\Psi_1(\lambda_1)$ are the wanted terms and contribute to the eigenvalue $\Lambda_L(\lambda|\{\lambda_1\})$. The remaining ones are the so-called unwanted terms and they have to be eliminated by imposing restrictions on the rapidity λ_1.

In this case it is easy to see that we have two unwanted terms which are directly eliminated by the condition $X_1(\lambda_1)=X_2(\lambda_1)$. However, the trace term on the right hand side of (5.4) does not give its wanted part directly.

By simple inspection of the $\mathcal{R}$-matrices (2.1) for the $\mathrm{C}^{(1)}_n$,$\mathrm{D}^{(1)}_n$ and $\mathrm{A}^{(2)}_{2n-1}$ vertex models, one can see that they trace in the auxiliary space is proportional to the N by N identity matrix $\mathbf{I}$

$$
\mathrm{Tr}_a[\mathcal{L}^{(1)}_{a1}(\lambda)]=\left(x^{(1)}_1(\lambda)+(N_1-2)x^{(1)}_2(\lambda)+y^{(1)}_{N_1N_1}(\lambda)\right)\mathbf{I},\qquad N_1=4,6,8,... \tag{5.5}
$$

It means that $\Psi_1(\lambda_1)$ is the eigenstate of $\tau_L(\lambda)$ with eigenvalue

$$
\begin{aligned}
\Lambda_L(\lambda|\{\lambda_1\}) &= X_1(\lambda)z(\lambda_1-\lambda)+X_3(\lambda)\frac{x_2(\lambda-\lambda_1)}{y_{NN}(\lambda-\lambda_1)} \\
&+X_2(\lambda)\left(\frac{X^{(1)}_1(\lambda)}{X^{(1)}_2(\lambda)}+(N_1-2)+\frac{X^{(1)}_3(\lambda)}{X^{(1)}_2(\lambda)}\right)
\end{aligned} \tag{5.6}
$$

provided that

$$
\frac{X_1(\lambda_1)}{X_2(\lambda_1)}=1 \tag{5.7}
$$

where we have used the notation

$$
X^{(1)}_3(\lambda)=y^{(1)}_{N_1N_1}(\lambda-\lambda_1),\qquad X^{(1)}_a(\lambda)=x^{(1)}_a(\lambda-\lambda_1),\quad a=1,2 \tag{5.8}
$$

Here we make some remarks in respect to (5.6). Although its computation is made on the ground ($l=0$), the result also contains Boltzmann weights of the model in the first *layer* ($l=1$). Of course, we can recall (4.21) to write (5.6) with all Boltzmann weights of the model on the ground. Nevertheless, as we will see later, this *layer* formation is very important -even when there is no nest to build. For instance, the eigenvalue (5.6) is not valid for the $\mathrm{D}^{(1)}_2$ vertex model because its first *layer* $l=1$ (the $\mathrm{D}^{(1)}_1$ model) is not regular. It is also curious to note that no further constraint is necessary for the vector $\mathcal{F}_1$.

However, when N_l is an odd number, the situation is not so simple because the $\mathrm{Tr}_a[\mathcal{L}^{(l)}_{a1}]$ is not more proportional to the identity matrix due to the weight $y^{(l)}_{\alpha\beta}(\lambda)$ with $\alpha=\beta$ and $\alpha=\alpha'$ (2.3), the element common to the diagonals of $\mathcal{R}^{(l)}$.

The trace in (5.4) represents the transfer matrix of $L+1$ site chain with one inhomogeneous site coming from the Lax operator $\mathcal{L}^{(1)}_{a1}(\lambda-\lambda_1)$. To solve the eigenvalue problem of $\tau^{(1)}_{L+1}(\lambda,\{\lambda_1\})\mathcal{F}_1\left|0_L\right\rangle$, we

first note that $\tau^{(1)}_{L+1} \in \mathcal{H}^{(L)} \otimes \mathcal{H}^{(1)}$ and the part of $\tau^{(1)}_{L+1}$ involving $\mathcal{D}(\lambda) \in \mathcal{H}^{(L)}$ commutes with $\mathcal{F}_1 \in \mathcal{H}^{(1)}$ and can hit directly the reference state $|0_L\rangle$

$$\tau^{(1)}_{L+1}(\lambda, \{\lambda_1\})\mathcal{F}_1 |0_L\rangle = X_2(\lambda) \left(\tau^{(1)}_1(\lambda, \{\lambda_1\})\mathcal{F}_1\right) |0_L\rangle . \tag{5.9}$$

It means that the eigenvalue condition for (5.4) leads to the requirement that $\mathcal{F}_1$ ought to be an eigenvavector of $\tau^{(1)}_1(\lambda, \{\lambda_1\})$. Therefore, if we suppose that

$$\tau^{(1)}_1(\lambda, \{\lambda_1\})\mathcal{F}_1 = \Lambda^{(1)}_1(\lambda, \{\lambda_1\}|\cdots)\mathcal{F}_1, \tag{5.10}$$

the last wanted term in (5.4) has the form

$$X_2(\lambda)\frac{\Lambda^{(1)}_1(\lambda, \{\lambda_1\}|\cdots)}{X^{(1)}_2(\lambda)}\mathcal{B}(\lambda_1)\mathcal{F}_1 |0_L\rangle \tag{5.11}$$

and we conclude for $\mathrm{B}^{(1)}_n$ and $\mathrm{A}^{(2)}_{2n}$ vertex models that $\Psi_1(\lambda_1)$ is an eigenfunction of $\tau_L(\lambda)$ with eigenvalue

$$\Lambda_L(\lambda|\{\lambda_1\}) = X_1(\lambda)z(\lambda_1 - \lambda) + X_2(\lambda)\frac{\Lambda^{(1)}_1(\lambda, \{\lambda_1\}|\cdots)}{X^{(1)}_2(\lambda)} + X_3(\lambda)\frac{x_2(\lambda - \lambda_1)}{y_{NN}(\lambda - \lambda_1)} \tag{5.12}$$

provided that

$$\frac{X_1(\lambda_1)}{X_2(\lambda_1)} = 1 \tag{5.13}$$

The eigenvalue (5.12) is partial because we still have to solve the eigenvalue problem (5.10) in order to know the value of $\Lambda^{(1)}_1(\lambda, \{\lambda_1\}|\cdots)$. Here we have reached a point which is typical of nested Bethe ansatz problems. It means that we have to solve an another eigenvalue problem for the transfer matrix $\tau^{(1)}_1$ with its $\mathcal{R}$-matrix given by (2.1) but with $l = 1$, i.e., the first *layer* of the nest.

The row-to-row 1-site inhomogeneous transfer matrix $\tau^{(1)}_1(\lambda, \{\lambda_1\})$ is given by

$$\tau^{(1)}_1(\lambda) = A^{(1)}_1(\lambda) + \sum_{\alpha=1}^{N_1-2} D^{(1)}_{\alpha\alpha}(\lambda) + A^{(1)}_1(\lambda) \tag{5.14}$$

The notation used in (5.14) is to be considered as a shorthand as these terms depend also on the inhomogeneous parameter λ_1. The reference state $|0_1\rangle^{(1)}$ is given by (5.9) and it is a highest vector

$$\begin{aligned}
A^{(1)}_1(\lambda)\,|0_1\rangle^{(1)} &= X^{(1)}_1(\lambda)|\,|0_1\rangle^{(1)}, \quad D^{(1)}_{\alpha\alpha}(\lambda)\,|0_1\rangle^{(1)} = X^{(1)}_2(\lambda)\,|0_1\rangle^{(1)}, \\
A^{(1)}_3(\lambda)\,|0_1\rangle^{(1)} &= X^{(1)}_3(\lambda)\,|0_1\rangle^{(1)}, \quad D^{(1)}_{\alpha\beta}(\lambda)\,|0_1\rangle^{(1)} = 0, \\
C^{(1)}_\alpha(\lambda)\,|0_1\rangle^{(1)} &= 0, \quad B^{(1)}_\alpha(\lambda)\,|0_1\rangle^{(1)} \neq \left\{0, |0_1\rangle^{(1)}\right\}, \quad \alpha \neq \beta = 1, 2, ..., N_1 - 2.
\end{aligned} \tag{5.15}$$

with eigenvalue

$$\Lambda^{(1)}_1(\lambda, \{\lambda_1\}|0) = X^{(1)}_1(\lambda) + (N_1 - 2)X^{(1)}_2(\lambda) + X^{(1)}_3(\lambda) \tag{5.16}$$

where $X^{(1)}_a(\lambda)$, $a = 1, 2, 3.$ are given by (5.8).

The condition that $\mathcal{F}_1$ ought to be an eigenvector of $\tau^{(1)}_1(\lambda)$ requires the diagonalization of $\tau^{(1)}_1(\lambda)$, which can be carried out by a second Bethe ansatz. There are two candidates for $\mathcal{F}_1 \in \mathcal{H}^{(1)}$: the own reference state $|0_1\rangle^{(1)}$ and the one-particle excitation $\mathcal{B}^{(1)}(\lambda^{(1)}_1)\,|0_1\rangle^{(1)}$.

The choice $\mathcal{F}_1 = |0_1\rangle^{(1)}$ implies in a particular linear combination for the one-particle state (5.1) and in this case $\Psi_1(\lambda_1)$ is an eigenfunction of $\tau_L(\lambda)$ for the $\mathrm{B}^{(1)}_n$ and $\mathrm{A}^{(2)}_{2n-1}$ with the eigenvalue expression

equal to (5.6) and the Bethe equation equal to (5.7). However, the second choice seems to be the most general but we will not consider it now. This case will be presented later in a more general context where the multi-particle state is treated.

Notice that we have not yet given an explicit rule to eliminate the unwanted terms, until now they were merely canceled out. In order to find such rule we will consider the two-particle state with more details.

6 The two-particle Bethe state

In analogy with the scalar case [23, 24] we have two types of linearly independent contributions $\mathcal{B}^{(l)}\otimes\mathcal{B}^{(l)}$ and B_{N_l-1} for the vector $\Phi_2^{(l)}(\lambda_1,\lambda_2)$. We seek vectors in the form

$$\Phi_2^{(l)}(\lambda_1,\lambda_2)=\mathcal{B}^{(l)}(\lambda_1)\otimes\mathcal{B}^{(l)}(\lambda_2)+B_{N_l-1}(\lambda_1)\Gamma(\lambda_1,\lambda_2) \tag{6.1}$$

where $\Gamma(\lambda_1,\lambda_2)$ is an operator-valued vector which has to be fixed such that $\Phi_2^{(l)}(\lambda_1,\lambda_2)$ is unique.

It was demonstrated in [23] that $\Phi_2^{(l)}(\lambda_1,\lambda_2)$ is unique provided it is ordered in a normal way: in general, the operator-valued vector $\Phi_m^{(l)}(\lambda_1,\dots,\lambda_m)$ is composite of normal-ordered monomials. A monomial is normally ordered if in it all elements $\mathcal{B}^{(l)}$, $\mathcal{B}^{*(l)}$ and B_{N_l-1} are on the left, and all elements $\mathcal{C}^{(l)}$, $\mathcal{C}^{*(l)}$ and C_{N_l-1} are on the right of all elements $A_1^{(l)}$, $\mathcal{D}^{(l)}$ and $A_3^{(l)}$. Moreover, all elements of one given type have standard ordering, i.e., $T^{(l)}_{\alpha_1\beta_1}(\lambda_1)T^{(l)}_{\alpha_2\beta_2}(\lambda_2)\cdots T^{(l)}_{\alpha_{mn}\beta_m}(\lambda_m)$.

Now we recall the intertwining relation (4.5) to solve the equations of the block $B[1,5]$ in order to get the following commutation relation

$$\begin{aligned}\mathcal{B}^{(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\mu) &= \left(\mathcal{B}^{(l)}(\mu)\otimes\mathcal{B}^{(l)}(\lambda)-\frac{\hat{Y}^{(l)}_{N_l2}(\mu-\lambda)}{y^{(l)}_{N_lN_l}(\mu-\lambda)}B_{N_l-1}(\mu)A_1^{(l)}(\lambda)\right)\frac{S^{(l+1)}(\lambda-\mu)}{x_1^{(l+1)}(\lambda-\mu)}\\ &\quad +B_{N_l-1}(\lambda)A_1^{(l)}(\mu)\frac{\hat{Y}_{N_l2}(\lambda-\mu)}{y_{N_lN_l}(\lambda-\mu)},\end{aligned} \tag{6.2}$$

from which we can see that (6.1) will be normally ordered if only if it satisfies the property

$$\Phi_2^{(l)}(\lambda_1,\lambda_2)=\Phi_2^{(l)}(\lambda_2,\lambda_1)\frac{S^{(l+1)}(\lambda_1-\lambda_2)}{x_1^{(l+1)}(\lambda_1-\lambda_2)} \tag{6.3}$$

This condition fixes $\Gamma(\lambda_1,\lambda_2)$ and our vector for the two-particle state has the form

$$\Phi_2^{(l)}(\lambda_1,\lambda_2)=\mathcal{B}^{(l)}(\lambda_1)\otimes\mathcal{B}^{(l)}(\lambda_2)+B_{N_l-1}(\lambda_1)A_1^{(l)}(\lambda_2)\frac{Y^{(l)}_{N_l2}(\lambda_1-\lambda_2)}{y^{(l)}_{N_lN_l}(\lambda_1-\lambda_2)} \tag{6.4}$$

Here we notice that the condition (6.3) must be generalized to include multi-particle state and it will play the main role in the elimination rules of the unwanted terms.

The action of the transfer matrix on this vector is more laborious. In addition to (4.12), (4.15) and (5.2) we appeal to (4.5) to derive nine further commutation relations.

Due to the presence of the scalar B_{N_l-1} in (6.4) we have solve the block $B[1,7]$ of (4.5) in order to get its commutation relations with A_1 and A_3 in a given *layer l*

$$\begin{aligned}A_1^{(l)}(\lambda)B_{N_l-1}(\mu) &= \frac{x_1^{(l)}(\mu-\lambda)}{y_{N_lN_l}(\mu-\lambda)}B_{N_l-1}(\mu)A_1^{(l)}(\lambda)-\frac{y^{(l)}_{1N_l}(\mu-\lambda)}{y^{(l)}_{N_lN_l}(\mu-\lambda)}B_{N_l-1}(\lambda)A_1^{(l)}(\mu)\\ &\quad -\mathcal{B}^{(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\mu)\frac{\hat{Y}^{(l)}_{2N_l}(\mu-\lambda)}{y^{(l)}_{N_lN_l}(\mu-\lambda)},\end{aligned} \tag{6.5}$$

and

$$A_3^{(l)}(\lambda)B_{N_l-1}(\mu) = \frac{x_1^{(l)}(\lambda-\mu)}{y_{N_lN_l}(\lambda-\mu)}B_{N_l-1}(\mu)A_3^{(l)}(\lambda) - \frac{y_{N_l1}^{(l)}(\lambda-\mu)}{y_{N_lN_l}^{(l)}(\lambda-\mu)}B_{N_l-1}(\lambda)A_3^{(l)}(\mu)$$
$$-\frac{\hat{Y}_{N_l2}^{(l)}(\lambda-\mu)}{y_{N_lN_l}^{(l)}(\lambda-\mu)}\mathcal{B}^{*(l)}(\lambda)\otimes\mathcal{B}^{*(l)}(\mu), \tag{6.6}$$

The block $B[2,6]$, $B[4,6]$ and $B[4,8]$ can be used to derive the relation

$$\mathcal{D}^{(l)}(\lambda)B_{N_l-1}(\mu) = z^{(l)}(\lambda-\mu)z^{(l)}(\mu-\lambda)B_{N_l-1}(\mu)\mathcal{D}^{(l)}(\lambda) + \frac{x_4^{(l)}(\lambda-\mu)^2}{x_2^{(l)}(\lambda-\mu)^2}B_{N_l-1}(\lambda)\mathcal{D}^{(l)}(\mu)$$
$$-\frac{x_4^{(l)}(\lambda-\mu)}{x_2^{(l)}(\lambda-\mu)}\left[\mathcal{B}^{(l)}(\lambda)\otimes\mathcal{B}^{*(l)}(\mu) - \mathcal{B}^{*(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\mu)\right]. \tag{6.7}$$

Again, we must work out (6.7 in order to get the commutation relations of B_{N_l-1} with the scalar $\mathrm{Tr}_a[\mathcal{D}^{(l)}(\lambda)]$

$$\mathrm{Tr}_a[\mathcal{D}^{(l)}(\lambda)]B_{N_l-1}(\mu) = z^{(l)}(\lambda-\mu)z^{(l)}(\mu-\lambda)B_{N-1}(\mu)\mathrm{Tr}_a[\mathcal{D}^{(l)}(\lambda)] + \frac{x_4^{(l)}(\lambda-\mu)^2}{x_2^{(l)}(\lambda-\mu)^2}B_{N-1}(\lambda)\mathrm{Tr}_a[\mathcal{D}^{(l)}(\mu)]$$
$$-\frac{x_4^{(l)}(\lambda-\mu)}{x_2^{(l)}(\lambda-\mu)}(\mathcal{B}^{(l)}(\lambda)\mathcal{B}^{*(l)}(\mu) - \mathrm{Tr}_a[\mathcal{B}^{*(l)}(\lambda)\mathcal{B}^{(l)}(\mu)]) \tag{6.8}$$

where we have used the identity

$$1 - \frac{x_4^{(l)}(\lambda-\mu)}{x_2^{(l)}(\lambda-\mu)}\frac{x_3^{(l)}(\lambda-\mu)}{x_2^{(l)}(\lambda-\mu)} = z^{(l)}(\lambda-\mu)z^{(l)}(\mu-\lambda) \tag{6.9}$$

Note also that these scalar relations will survive the reduction mechanism for the six-vertex models previously presented.

Since we need a second commutation step in order to that $\tau_L^{(l)}$ hits the reference state, we compute the commutation relations among creation and annihilation operators

$$\mathcal{C}^{(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\mu) = \mathcal{B}^{(l)}(\mu)\otimes\mathcal{C}^{(l)}(\lambda) - \frac{x_4^{(l)}(\lambda-\mu)}{x_2^{(l)}(\lambda-\mu)}\left[A_1^{(l)}(\lambda)D^{(l)}(\mu) - A_1^{(l)}(\mu)\mathcal{D}^{(l)}(\lambda))\right] \tag{6.10}$$

$$\mathcal{C}^{(l)}(\lambda)B_{N_l-1}(\mu) = \frac{x_2^{(l)}(\mu-\lambda)}{y_{N_lN_l}^{(l)}(\mu-\lambda)}B_{N_l-1}(\mu)\mathcal{C}^{(l)}(\lambda) + \frac{x_3^{(l)}(\mu-\lambda)}{y_{N_lN_l}^{(l)}(\mu-\lambda)}\mathcal{B}^{*(l)}(\lambda)A_1^{(l)}(\mu)$$
$$-\frac{y_{1N_l}^{(l)}(\mu-\lambda)}{y_{N_lN_l}^{(l)}(\mu-\lambda)}\mathcal{B}^{*(l)}(\mu)A_1^{(l)}(\lambda) - \mathcal{D}^{(l)}\otimes\mathcal{B}^{(l)}(\lambda)C_{N_l-1}(\mu)\frac{\hat{Y}_{2N_l}^{(l)}(\mu-\lambda)}{y_{N_lN_l}^{(l)}(\mu-\lambda)}, \tag{6.11}$$

both relations were obtained from the blocks $B[2,2]$ and $B[4,7]$, respectively. We also need the commutation of $B_{N_l-1}(\lambda)$ with the other creation operators

$$B_{N_l-1}(\lambda)\mathcal{B}^{(l)}(\mu) = z^{(l)}(\mu-\lambda)\mathcal{B}^{(l)}(\mu)B_{N_l-1}(\lambda) - \frac{x_4^{(l)}(\mu-\lambda)}{x_2^{(l)}(\mu-\lambda)}\mathcal{B}^{(l)}(\lambda)B_{N_l-1}(\mu), \tag{6.12}$$

and

$$\mathcal{B}^{*(l)}(\lambda)B_{N_l-1}(\mu)=z^{(l)}(\lambda-\mu)B_{N_l-1}(\mu)\mathcal{B}^{*(l)}(\lambda)-\frac{x_4^{(l)}(\lambda-\mu)}{x_2^{(l)}(\lambda-\mu)}B_{N_l-1}(\lambda)\mathcal{B}^{*(l)}(\mu). \tag{6.13}$$

which are obtained from the blocks $B[1,6]$, $B[1,8]$ and $B[1,10]$.

Here we observe that the final action of $\tau_L^{(l)}(\lambda)$ on normally ordered vectors must be normal ordered. This implies an increasing use of commutation relations needed for the eigenvalue problem. For instance, the action of the scalar $A_1^{(l)}(\lambda)$ on the vector $\Phi_2^{(l)}(\lambda_1,\lambda_2)$ has its normal ordered form given by

$$\begin{aligned}
A_1^{(l)}(\lambda)\Phi_2^{(l)}(\lambda_1,\lambda_2) \;=\;& \prod_{k=1}^{2} z^{(l)}(\lambda_k-\lambda)\Phi_2^{(l)}(\lambda_1,\lambda_2)A_1^{(l)}(\lambda) \\
&-\frac{x_3^{(l)}(\lambda_1-\lambda)}{x_2^{(l)}(\lambda_1-\lambda)}z^{(l)}(\lambda_{21})\mathcal{B}^{(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\lambda_2)A_1^{(l)}(\lambda_1) \\
&-\frac{x_3^{(l)}(\lambda_2-\lambda)}{x_2^{(l)}(\lambda_2-\lambda)}z^{(l)}(\lambda_{12})\mathcal{B}^{(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\lambda_1)A_1^{(l)}(\lambda_2)\frac{S_{12}^{(l+1)}(\lambda_1-\lambda_2)}{x_1^{(l+1)}(\lambda_1-\lambda_2)} \\
&+B_{N_l-1}(\lambda)\mathbf{G}_{21}^{(l)}(\lambda,\lambda_1,\lambda_2)A_1^{(l)}(\lambda_1)A_1^{(l)}(\lambda_2)+\cdots
\end{aligned} \tag{6.14}$$

where we have defined a row vector with N_{l+1}^2 entries

$$\mathbf{G}_{21}^{(l)}(\lambda,\lambda_1,\lambda_2)=\frac{x_3^{(l)}(\lambda_2-\lambda)}{x_2^{(l)}(\lambda_2-\lambda)}\frac{\hat{Y}_{N_l\ 2}^{(l)}(\lambda-\lambda_1)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_1)}\frac{S^{(l+1)}(\lambda_1-\lambda)}{x_2^{(l+1)}(\lambda_1-\lambda)}+\frac{y_{1N_l}^{(l)}(\lambda_1-\lambda)}{y_{N_lN_l}^{(l)}(\lambda_1-\lambda)}\frac{\hat{Y}_{N_l\ 2}^{(l)}(\lambda_1-\lambda_2)}{y_{N_lN_l}^{(l)}(\lambda_1-\lambda_2)} \tag{6.15}$$

which satisfies the cyclic permutation property

$$\mathbf{G}_{21}^{(l)}(\lambda,\lambda_1,\lambda_2)=\mathbf{G}_{21}^{(l)}(\lambda,\lambda_2,\lambda_1)\frac{S^{(l+1)}(\lambda_1-\lambda_2)}{x_1^{(l+1)}(\lambda_1-\lambda_2)} \tag{6.16}$$

Observe the mix of *layer*s with respect the Boltzmann weights of $\mathbf{G}_{21}^{(l)}$ which really acts on the $l+1$ *layer*.

The action of the scalar $A_3^{(l)}(\lambda)$ on the vector $\Phi_2^{(l)}(\lambda_1,\lambda_2)$ has a very similar normal ordered form

$$\begin{aligned}
A_3^{(l)}(\lambda)\Phi_2^{(l)}(\lambda_1,\lambda_2) \;=\;& \prod_{k=1}^{2}\frac{x_2^{(l)}(\lambda-\lambda_k)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_k)}\ \Phi_2^{(l)}(\lambda_1,\lambda_2)A_3^{(l)}(\lambda) \\
&-z^{(l)}(\lambda_{21})\frac{\hat{Y}_{N_l2}^{(l)}(\lambda-\lambda_2)}{y_{N_lNl}(\lambda-\lambda_2)}\ \mathcal{B}^{*(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\lambda_1)\otimes\mathcal{D}^{(l)}(\lambda_2) \\
&-z^{(l)}(\lambda_{12})\frac{\hat{Y}_{N_l2}^{(l)}(\lambda-\lambda_1)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_1)}\ \mathcal{B}^{*(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\lambda_2)\otimes\mathcal{D}^{(l)}(\lambda_1)\frac{S_{12}^{(l+1)}(\lambda_1-\lambda_2)}{x_1^{(l+1)}(\lambda_1-\lambda_2)} \\
&+B_{N_l-1}(\lambda)\mathbf{H}_{21}^{(l)}(\lambda,\lambda_1,\lambda_2)\mathcal{D}^{(l)}(\lambda_1)\otimes\mathcal{D}^{(l)}(\lambda_2)+\cdots
\end{aligned} \tag{6.17}$$

where the vector $\mathbf{H}_{21}^{(l)}$ is given by

$$\mathbf{H}_{21}^{(l)}(\lambda,\lambda_1,\lambda_2)=\frac{y_{N_l1}^{(l)}(\lambda-\lambda_1)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_1)}\frac{\hat{Y}_{N_l\ 2}^{(l)}(\lambda_1-\lambda_2)}{y_{N_l\ N_l}^{(l)}(\lambda_1-\lambda_2)}-\frac{x_3^{(l)}(\lambda-\lambda_1)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_1)}\frac{\hat{Y}_{N_l2}^{(l)}(\lambda-\lambda_2)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_2)} \tag{6.18}$$

and satisfies the cyclic permutation property

$$\mathbf{H}_{21}^{(l)}(\lambda,\lambda_1,\lambda_2)=\mathbf{H}_{21}^{(l)}(\lambda,\lambda_2,\lambda_1)\frac{S^{(l+1)}(\lambda_1-\lambda_2)}{x_1^{(l+1)}(\lambda_1-\lambda_2)}. \tag{6.19}$$

Finally, the action of the scalar $\mathrm{Tr}_a[\mathcal{D}^{(l)}(\lambda)]$ on the vector $\Phi_2^{(l)}(\lambda_1,\lambda_2)$ is a little bit different

$$\begin{aligned}
&\mathrm{Tr}_a[\mathcal{D}^{(l)}(\lambda)]\Phi_2^{(l)}(\lambda_1,\lambda_2) = \Phi_2^{(l)}(\lambda_1,\lambda_2)\mathrm{Tr}_a[\frac{\mathcal{L}_{a2}^{(l+1)}(\lambda-\lambda_2)}{x_2^{(l+1)}(\lambda-\lambda_2)}\frac{\mathcal{L}_{a1}^{(l+1)}(\lambda-\lambda_1)}{x_2^{(l+1)}(\lambda-\lambda_{1)}}\mathcal{D}^{(l)}(\lambda)] \\
&-z^{(l)}(\lambda_1-\lambda_2)\frac{x_4^{(l)}(\lambda-\lambda_1)}{x_2^{(l)}(\lambda-\lambda_1)}B^{(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\lambda_2)[\mathcal{D}^{(l)}(\lambda_1)\otimes\mathbf{1}^{(l)}]\frac{\mathcal{R}^{(l+1)}(\lambda_1-\lambda_2)}{x_1^{(l+1)}(\lambda_1-\lambda_2)} \\
&-z^{(l)}(\lambda_2-\lambda_1)\frac{x_4^{(l)}(\lambda-\lambda_2)}{x_2^{(l)}(\lambda-\lambda_2)}B^{(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\lambda_1)[\mathcal{D}^{(l)}(\lambda_2)\otimes\mathbf{1}^{(l)}] \\
&+z^{(l)}(\lambda_2-\lambda_1)\frac{\hat{Y}_{N_l\ 2}^{(l)}(\lambda-\lambda_1)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_1)}[B^{*(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\lambda_2)\otimes\mathbf{1}^{(l)}]A_1^{(l)}(\lambda_1) \\
&+z^{(l)}(\lambda_1-\lambda_2)\frac{\hat{Y}_{N_l\ 2}^{(l)}(\lambda-\lambda_2)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_2)}[B^{*(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\lambda_1)\otimes\mathbf{1}^{(l)}]\frac{\mathcal{R}_{12}^{(l+1)}(\lambda_1-\lambda_2)}{x_1^{(l+1)}(\lambda_1-\lambda_2)}A_1^{(l)}(\lambda_2) \\
&+B_{N_l-1}(\lambda)\mathbf{Y}_{21}^{(l)}(\lambda,\lambda_1,\lambda_2)[\mathcal{D}^{(l)}(\lambda_2)\otimes\mathbf{1}^{(l)}]A_1^{(l)}(\lambda_1) \\
&+B_{N_l-1}(\lambda)\mathbf{Y}_{21}^{(l)}(\lambda,\lambda_2,\lambda_1)[\mathcal{D}^{(l)}(\lambda_1)\otimes\mathbf{1}^{(l)}]A_1^{(l)}(\lambda_2)\frac{\mathcal{R}_{12}^{(l+1)}(\lambda_1-\lambda_2)}{x_1^{(l+1)}(\lambda_1-\lambda_2)}+\cdots
\end{aligned} \tag{6.20}$$

where the vector $\mathbf{Y}_{21}^{(l)}$ is given by

$$\mathbf{Y}_{21}^{(l)}(\lambda,\lambda_1,\lambda_2) = [z^{(l)}(\lambda-\lambda_1)\frac{x_4^{(l)}(\lambda-\lambda_2)}{x_2^{(l)}(\lambda-\lambda_2)} - \frac{x_4^{(l)}(\lambda-\lambda_1)}{x_2^{(l)}(\lambda-\lambda_1)}\frac{x_4^{(l)}(\lambda_1-\lambda_2)}{x_2^{(l)}(\lambda_1-\lambda_2)}]\frac{\hat{Y}_{N_l2}^{(l)}(\lambda-\lambda_1)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_1)} \tag{6.21}$$

and also satisfies the cyclic permutation property

$$\mathbf{Y}_{21}^{(l)}(\lambda,\lambda_1,\lambda_2) = \mathbf{Y}_{21}^{(l)}(\lambda,\lambda_2,\lambda_1)\frac{S^{(l+1)}(\lambda_1-\lambda_2)}{x_1^{(l+1)}(\lambda_1-\lambda_2)} \tag{6.22}$$

The ellipses in these expressions denote normally ordered terms containing factors of the type $\mathcal{C}^{(l)}$, $\mathcal{C}^{*(l}$ and C_{N_l-1}.

It should be worth note that we have used some matrix identities to derive (6.14)-(6.20)

$$\begin{aligned}
\frac{S^{(l+1)}(\lambda_{ab})}{x_1^{(l+1)}(\lambda_{ab})}\frac{S^{(l+1)}(\lambda_{ba})}{x_1^{(l+1)}(\lambda_{ba})} &= I, \\
\frac{x_3^{(l)}(\lambda_{cb})}{x_2^{(l)}(\lambda_{cb})}\frac{S^{(l+1)}(\lambda_{ab})}{x_2^{(l+1)}(\lambda_{ab})} - \frac{\hat{Y}_{2N}^{(l)}(\lambda_{ab})}{y_{NN}^{(l)}(\lambda_{ab})}\frac{\hat{Y}_{N2}^{(l)}(\lambda_{ac})}{y_{NN}^{(l)}(\lambda_{ac})} &= \frac{x_3^{(l)}(\lambda_{ab})}{x_2^{(l)}(\lambda_{ab})}\frac{x_3^{(l)}(\lambda_{ca})}{x_2^{(l)}(\lambda_{ca})}I + \frac{x_3^{(l)}(\lambda_{cb})}{x_2^{(l)}(\lambda_{cb})}\frac{S^{(l+1)}(\lambda_{ac})}{x_2^{(l+1)}(\lambda_{ac})} \\
\frac{x_2^{(l)}(\lambda_{ab})}{y_{NN}^{(l)}(\lambda_{ab})}\frac{\hat{Y}_{N_l2}^{(l)}(\lambda_{ac})}{y_{N_lN_l}^{(l)}(\lambda_{ac})} + \frac{x_3^{(l)}(\lambda_{cb})}{x_2^{(l)}(\lambda_{cb})}\frac{\hat{Y}_{N_l2}^{(l)}(\lambda_{ab})}{y_{N_lN_l}^{(l)}(\lambda_{ab})} &= z^{(l)}(\lambda_{cb})\frac{\hat{Y}_{N_l2}^{(l)}(\lambda_{ac})}{y_{N_lN_l}^{(l)}(\lambda_{ac})}, \qquad (a\neq b\neq c)
\end{aligned} \tag{6.23}$$

where I is a N_{l+1}^2 by N_{l+1}^2 matrix identity and $\lambda_{ab}=\lambda_a-\lambda_b$.

Let us begin with the eigenvalue problem for the two-particle state in a homogeneous L site lattice which is defined by the linear combination

$$\Psi_2(\lambda_1,\lambda_2) = \Phi_2(\lambda_1,\lambda_2)\mathcal{F}_2\,|0_L\rangle \tag{6.24}$$

where Φ_2 is given by (6.4) and the vector $\mathcal{F}_2$ has components $f^{\alpha\beta}\in C$, $(\alpha,\beta=1,...,N_{l+1})$.

The action of $\tau_L(\lambda)$ on $\Psi_2(\lambda_1,\lambda_2)$ can now be computed using (6.14), (6.17) and (6.20):

$$
\begin{aligned}
\tau_L(\lambda)\Psi_2(\lambda_1,\lambda_2) \;=\;& X_1(\lambda)\prod_{k=1}^{2} z(\lambda_k-\lambda)\Psi_2(\lambda_1,\lambda_2) + X_3(\lambda)\prod_{k=1}^{2}\frac{x_2(\lambda-\lambda_k)}{y_{NN}(\lambda-\lambda_k)}\;\Psi_2(\lambda_1,\lambda_2) \\
& + X_2(\lambda)\Phi_2(\lambda_1,\lambda_2)\frac{1}{X_2^{(1)}(\lambda)}\left\{\tau_2^{(1)}(\lambda)\mathcal{F}_2\right\}|0_L\rangle \\
& - \frac{x_3(\lambda_{10})}{x_2(\lambda_{10})}\mathcal{B}(\lambda)\otimes\Phi_1(\lambda_2)[X_1(\lambda_1)z(\lambda_{21}) - X_2(\lambda_1)z(\lambda_{12})\frac{\mathcal{R}^{(1)}(\lambda_{12})}{x_1^{(1)}(\lambda_{12})}]\mathcal{F}_2\,|0_L\rangle \\
& - \frac{x_3(\lambda_{20})}{x_2(\lambda_{20})}\mathcal{B}(\lambda)\otimes\Phi_1(\lambda_1)[X_1(\lambda_2)z(\lambda_{12})\frac{S_{12}^{(1)}(\lambda_{12})}{x_1^{(1)}(\lambda_{12})} - X_2(\lambda_2)z(\lambda_{21})]\mathcal{F}_2\,|0_L\rangle \\
& - \frac{\hat{Y}_{N2}(\lambda_{02})}{y_{NN}(\lambda_{02})}\;\mathcal{B}^*(\lambda)\otimes\Phi_1(\lambda_1)\otimes 1[X_2(\lambda_2)z(\lambda_{21}) - X_1(\lambda_2)z(\lambda_{12})\frac{\mathcal{R}_{12}^{(1)}(\lambda_{12})}{x_1^{(1)}(\lambda_{12})}]\mathcal{F}_2\,|0_L\rangle \\
& - \frac{\hat{Y}_{N2}(\lambda_{01})}{y_{NN}(\lambda_{01})}\;\mathcal{B}^*(\lambda)\otimes\Phi_1(\lambda_2)\otimes 1[X_2(\lambda_1)z(\lambda_{12})\frac{S_{12}^{(1)}(\lambda_{12})}{x_1^{(1)}(\lambda_{12})} - X_1(\lambda_1)z(\lambda_{21})]\mathcal{F}_2\,|0_L\rangle \\
& + B_{N-1}(\lambda)\{\mathbf{G}_{21}(\lambda,\lambda_1,\lambda_2)X_1(\lambda_1)X_1(\lambda_2) + \mathbf{H}_{21}(\lambda,\lambda_1,\lambda_2)X_2(\lambda_1)X_2(\lambda_2) \\
& + \mathbf{Y}_{21}(\lambda,\lambda_1,\lambda_2)X_2(\lambda_2)X_1(\lambda_1) + \mathbf{Y}_{21}(\lambda,\lambda_2,\lambda_1)X_2(\lambda_1)X_1(\lambda_2)\frac{\mathcal{R}_{12}^{(1)}(\lambda_{12})}{x_1^{(1)}(\lambda_{12})}\}\mathcal{F}_2\,|0_L\rangle
\end{aligned}
\tag{6.25}
$$

where we have used the rapidity difference notation with $\lambda_0 = \lambda$ and the definitions

$$
\Phi_1(\lambda_i) = \mathcal{B}(\lambda_i), \qquad X_i^{(1)}(\lambda) = \prod_{k=1}^{2} x_i^{(1)}(\lambda-\lambda_k) \qquad (i=1,2) \tag{6.26}
$$

Moreover, the unwanted terms were combined and we are presenting the two-site inhomogeneous transfer matrix for the first *layer*

$$
\tau_2^{(1)}(\lambda) = \mathrm{Tr}_a\left[\mathcal{L}_{a2}^{(1)}(\lambda-\lambda_2)\mathcal{L}_{a1}^{(1)}(\lambda-\lambda_1)\right]. \tag{6.27}
$$

Before we see how the unwanted terms can be canceled, let us first to consider the eigenvalue problem

$$
\tau_2^{(1)}(\lambda)\mathcal{F}_2 = \Lambda_2^{(1)}(\lambda,\{\lambda_i\}|\{\lambda_i^{(1)}\})\mathcal{F}_2 \qquad (i=1,2), \tag{6.28}
$$

where our choice for the vector $\mathcal{F}_2$ is implicit

$$
\mathcal{F}_2 = \Phi_2^{(1)}(\lambda_1^{(1)},\lambda_2^{(1)})\,|0_2\rangle^{(1)}. \tag{6.29}
$$

and above $\Phi_2^{(1)}$ is given by (6.4) and $|0_2\rangle^{(1)}$ by (3.8). From these results we have an incomplete expression for the eigenvalue

$$
\Lambda_L(\lambda|\{\lambda_1,\lambda_2\}) = X_1(\lambda)\prod_{k=1}^{2} z(\lambda_k-\lambda) + X_3(\lambda)\prod_{k=1}^{2}\frac{x_2(\lambda-\lambda_k)}{y_{NN}(\lambda-\lambda_k)} + X_2(\lambda)\frac{\Lambda_2^{(1)}(\lambda,\{\lambda_i\}|\{\lambda_i^{(1)}\})}{X_2^{(1)}(\lambda)} \tag{6.30}
$$

because $\Lambda_2^{(1)}(\lambda,\{\lambda_i\}|\{\lambda_i^{(1)}\})$ is still unknown.

At this point it is worth giving further information about the shorthand notation used in this text:

- $\tau_L(\lambda)$ is a row-to-row homogeneous transfer matrix in a L site lattice and $\Lambda_m(\lambda|\{\lambda_i\})$ is the eigenvalue of $\tau_L(\lambda)$ for the m-particle state with rapidities λ_i, $i=1,2,...m$. Acting with $\tau_L(\lambda)$ on its reference state $|0_L\rangle$ we have the factors

$$X_a(\lambda)=[x_a(\lambda)]^L, \quad a=1,2 \qquad \text{and} \qquad X_3(\lambda)=[y_{NN}(\lambda)]^L \tag{6.31}$$

- $\tau_m^{(l)}(\lambda) \mathrel{\mathring{=}} \tau_m^{(l)}(\lambda,\{\lambda_i^{(l-1)}\})$ is a row-to-row inhomogeneous transfer matrix in a m site lattice with inhomogeneous parameters $\lambda_i^{(l-1)}, i=1,2,...,m$. Its eigenvalue for the m-particle state is denoted by $\Lambda_m^{(l)}(\lambda,\{\lambda_i^{(l-1)}\}|\{\lambda_i^{(l)}\})$ where $\lambda_i^{(l)}$ are rapidities of the particles and $\lambda_i^{(l-1)}$ are inhomogeneous parameters of the corresponding lattice. Acting with $\tau_m^{(l)}(\lambda)$ with $l\geq 1$ on its reference state $|0_m\rangle^{(l)}$ we have the factors

$$\begin{aligned} X_a^{(l)}(\lambda) &\mathrel{\mathring{=}} X_a^{(l)}(\lambda,\{\lambda_i^{(l-1)}\})=\prod_{k=1}^{m} x_a^{(l)}(\lambda-\lambda_k^{(l-1)}), \quad a=1,2 \\ X_3^{(l)}(\lambda) &\mathrel{\mathring{=}} X_3^{(l)}(\lambda,\{\lambda_i^{(l-1)}\})=\prod_{k=1}^{m} y_{N_lN_l}^{(l)}(\lambda-\lambda_k^{(l-1)}), \quad i=1,...,m. \end{aligned} \tag{6.32}$$

  where m is the particle (site) number in the *layer* l and $\lambda_k^{(0)}=\lambda_k$.

In respect to the unwanted terms of (6.25), the main motivation to write this section, we first recall the relation (6.3) in its generalized form by using the cyclic permutations of the factors in the normal ordered vector for the m-particle state in the l-*th layer*

$$\Phi_m^{(l)}(\lambda_1,\lambda_2,...,\lambda_m)=\Phi_m^{(l)}(\lambda_2,...,\lambda_m,\lambda_1)\frac{S_{12}^{(l+1)}(\lambda_1-\lambda_2)}{x_1^{(l+1)}(\lambda_1-\lambda_2)}\frac{S_{23}^{(l+1)}(\lambda_1-\lambda_3)}{x_1^{(l+1)}(\lambda_1-\lambda_3)}\cdots\frac{S_{m-1,m}^{(l+1)}(\lambda_1-\lambda_m)}{x_1^{(l+1)}(\lambda_1-\lambda_m)} \tag{6.33}$$

Now we define an operator M by

$$M_m^{(l+1)}(\lambda,\{\lambda_i\})=\mathrm{Tr}_a\left[\frac{S_{a1}^{(l+1)}(\lambda_1-\lambda_2)}{x_1^{(l+1)}(\lambda_1-\lambda_2)}\frac{S_{a2}^{(l+1)}(\lambda_1-\lambda_3)}{x_1^{(l+1)}(\lambda_1-\lambda_3)}\cdots\frac{S_{a,m}^{(l+1)}(\lambda_1-\lambda_m)}{x_1^{(l+1)}(\lambda_1-\lambda_m)}\right] \tag{6.34}$$

and we remark that M is the normalized permutation of a m site inhomogeneous transfer matrix

$$\tau_m^{(l+1)}(\lambda,\{\lambda_i\})=\mathrm{Tr}_a\left[\mathcal{L}_{am}^{(l+1)}(\lambda-\lambda_m)\mathcal{L}_{am-1}^{(l+1)}(\lambda-\lambda_{m-1})\cdots\mathcal{L}_{a1}^{(l+1)}(\lambda-\lambda_1)\right] \tag{6.35}$$

Thus, we can write the generalization of (6.3) in a more convenient form

$$\Phi_m^{(l)}(\lambda_1,\lambda_2,...,\lambda_m)=\Phi_m^{(l)}(\lambda_2,...,\lambda_m,\lambda_1)M_m^{(l+1)}(\lambda_1,\{\lambda_i\}) \tag{6.36}$$

where $i=1,...,m$.

Using these expressions with $l=1$ and $m=2$, one can see that the unwanted term $\mathcal{B}(\lambda)\otimes\Phi_1(\lambda_2)$ in (6.25) can be eliminated by the condition

$$\left(z(\lambda_2-\lambda_1)X_1(\lambda_1)-z(\lambda_1-\lambda_2)X_2(\lambda_1)\frac{\tau_2^{(1)}(\lambda_1)}{x_1^{(1)}(\lambda_1-\lambda_2)}\right)\mathcal{F}_2=0 \tag{6.37}$$

and from the fact that $\mathcal{B}(\lambda)\otimes\Phi_1(\lambda_1)$ is a permutation of $\mathcal{B}(\lambda)\otimes\Phi_1(\lambda_2)$ by the interchange $\lambda_1\leftrightarrow\lambda_2$, one can use (6.36) with $l=1$ and $m=2$, in order to get a second elimination condition

$$M_2^{(1)}(\lambda_1,\{\lambda_i\})\left(z(\lambda_1-\lambda_2)X_1(\lambda_2)-z(\lambda_2-\lambda_1)X_2(\lambda_2)\frac{\tau_2^{(1)}(\lambda_2)}{x_1^{(1)}(\lambda_2-\lambda_1)}\right)\mathcal{F}_2=0 \qquad (6.38)$$

Consequently, we have the following eigenvalue problems

$$\tau_2^{(1)}(\lambda_a)\mathcal{F}_2=x_1^{(1)}(\lambda_a-\lambda_b)\frac{z(\lambda_b-\lambda_a)}{z(\lambda_a-\lambda_b)}\frac{X_1(\lambda_a)}{X_2(\lambda_a)}\mathcal{F}_2 \qquad (a\neq b=1,2) \qquad (6.39)$$

which should be seen as a generalization of the Bethe equations since this restriction alone eliminate all unwanted terms. To see this, one can use the same statements in $\mathcal{B}^*(\lambda)\otimes\Phi_1(\lambda_1)\otimes 1$ and $\mathcal{B}^*(\lambda)\otimes\Phi_1(\lambda_2)\otimes 1$ terms of (6.25) in order to get the same pair of equations (6.39). Finally, substituting (6.39) into the $B_{N-1}(\lambda)$ unwanted term of (6.25) one can see that it is also canceled out.

It is now clear that the eigenvalue (6.30) as well as the Bethe equations are written in terms of the eigenvalue of $\tau_2^{(1)}(\lambda)$:

$$\frac{X_1(\lambda_a)}{X_2(\lambda_a)}=\frac{z(\lambda_a-\lambda_b)}{z(\lambda_b-\lambda_a)}\frac{\Lambda_2^{(1)}(\lambda_a,\{\lambda_i\}|\{\lambda_i^{(1)}\})}{x_1^{(1)}(\lambda_a-\lambda_b)} \qquad (a\neq b=1,2) \qquad (6.40)$$

where $\Lambda_2^{(1)}(\lambda_a,\{\lambda_i\}|\{\lambda_i^{(1)}\})$ is the residue of $\Lambda_2^{(1)}(\lambda,\{\lambda_i\}|\{\lambda_i^{(1)}\})$ at $\lambda=\lambda_a$. Consequently, we have to consider this second eigenvalue problem (6.28) in order to fix the results of the first eigenvalue problem. Indeed it is a simple task because all computation was already made and the result follows directly read from (6.25) with the following trivial modifications:

- Changing the vertex models: $(\mathrm{B}_n^{(1)},\mathrm{C}_n^{(1)},\mathrm{D}_n^{(1)},\mathrm{A}_{2n}^{(2)},\mathrm{A}_{2n-1}^{(2)})\rightarrow(\mathrm{B}_{n-1}^{(1)},\mathrm{C}_{n-1}^{(1)},\mathrm{D}_{n-1}^{(1)},\mathrm{A}_{2(n-1)}^{(2)},\mathrm{A}_{2(n-1)-1}^{(2)})$, putting the label $l=1$ in the new Boltzmann weights.
- Changing the lattice: L site homogeneous lattice $(\tau_L(\lambda))\rightarrow 2$ -site inhomogeneous lattice $(\tau_2^{(1)}(\lambda))$.
- Changing the two-particle state: $\Phi_2(\lambda_1,\lambda_2)\mathcal{F}_2\left|0_L\right\rangle\rightarrow\Phi_2^{(1)}(\lambda_1^{(1)},\lambda_2^{(1)})\mathcal{F}_2^{(1)}\left|0_2\right\rangle^{(1)}$.

After these modifications we still have to consider a third eigenvalue problem by repeating this changing procedure by increasing by one unity the value of the label l but keep the size of the last lattice. Next, we repeat the last procedure and so on. Of course, such a procedure must have an end.

In the nested Bethe ansatz procedure the end is obtained in the *layer* for which the eigenvalue and the Bethe equations are fixed. It means that we are work with the models in the last *layer*.

The value $l=n-1$ defines the last *layer* as being the $\mathrm{B}_1^{(1)}$ and $\mathrm{A}_2^{(2)}$ vertex models (N_l odd) and as the $\mathrm{C}_2^{(1)}$,$\mathrm{D}_2^{(1)}$.an $\mathrm{A}_3^{(2)}$ vertex models (N_l even). Thus they must be considered separately:

6.1 $\mathrm{B}_n^{(1)}$ and $\mathrm{A}_{2n}^{(2)}$ two-particle state

We already mentioned that these models possess a limit via the reduction to scalar status. It means that in the last *layer* we are working with (4.6) and (4.7) with the Boltzmann weights of the $\mathrm{B}_1^{(1)}$ and $\mathrm{A}_2^{(2)}$ vertex models. The explicit expression for the eigenvalue problem of the L site homogeneous transfer matrix for these nineteen-vetex models was already presented in the reference [24]. However, here this expression will be presented by a reduction procedure from our general formulation.

Consequently the eigenvalue expression (6.25) is reduced to

$$
\begin{aligned}
\tau_2(\lambda)\Psi_2(\lambda_1,\lambda_2) \quad = \quad & (X_1(\lambda)\prod_{k=1}^{2} z(\lambda_k-\lambda) + X_2(\lambda)\prod_{k=1}^{2}\frac{z(\lambda-\lambda_k)}{\omega(\lambda-\lambda_k)} + X_3(\lambda)\prod_{k=1}^{2}\frac{x_2(\lambda-\lambda_k)}{y_{33}(\lambda-\lambda_k)})\Psi_2(\lambda_1,\lambda_2) \\
& -\frac{x_3(\lambda_{10})}{x_2(\lambda_{10})}\left(X_1(\lambda_1)z(\lambda_{21}) - X_2(\lambda_1)\frac{z(\lambda_{12})}{\omega(\lambda_{12})}\right)B_1(\lambda)B_1(\lambda_2)\left|0_2\right\rangle \\
& -\frac{x_3(\lambda_{20})}{x_2(\lambda_{20})}\left(X_1(\lambda_2)\frac{z(\lambda_{12})}{\omega(\lambda_{12})} - X_2(\lambda_2)z(\lambda_{21})\right)B_1(\lambda)B_1(\lambda_2)\left|0_2\right\rangle \\
& -\frac{y_{32}(\lambda_{02})}{y_{33}(\lambda_{02})}\left(X_2(\lambda_2)z(\lambda_{21}) - X_1(\lambda_2)\frac{z(\lambda_{12})}{\omega(\lambda_{12})}\right)B_3(\lambda)B_1(\lambda_1)\left|0_2\right\rangle \\
& -\frac{y_{32}(\lambda_{01})}{y_{33}(\lambda_{01})}\left(X_2(\lambda_1)\frac{z(\lambda_{12})}{\omega(\lambda_{12})} - X_1(\lambda_1)z(\lambda_{21})\right)B_3(\lambda)B_1(\lambda_2)\left|0_2\right\rangle \\
& +B_2(\lambda)\left\{G_{21}(\lambda,\lambda_1,\lambda_2)X_1(\lambda_1)X_1(\lambda_2) + H_{21}(\lambda,\lambda_1,\lambda_2)X_2(\lambda_1)X_2(\lambda_2)\right. \\
& \left. + Y_{21}(\lambda,\lambda_1,\lambda_2)X_2(\lambda_2)X_1(\lambda_1) + Y_{21}(\lambda,\lambda_2,\lambda_1)X_2(\lambda_1)X_1(\lambda_2)\frac{1}{\omega(\lambda_{12})}\right\}\left|0_2\right\rangle \qquad (6.41)
\end{aligned}
$$

where we have omitted the label $l = n-1$.

In (6.41) we have used the function $\omega(\lambda)$ which is the reduction of the objects S, $\mathcal{R}$ and $\mathrm{Tr}_a[\mathcal{L}]$ present in (6.25)

$$
\begin{aligned}
\frac{1}{\omega(\lambda)} \quad &= \quad \frac{s(\lambda)}{x_1(\lambda)} = \frac{\mathcal{R}(\lambda)}{x_1(\lambda)} = \frac{\mathrm{Tr}_a[\mathcal{L}(\lambda)]}{x_1(\lambda)} = \frac{1}{x_1(\lambda)}\left(y_{22}(\lambda) - \frac{y_{23}(\lambda)y_{32}(\lambda)}{y_{33}(\lambda)}\right). \\
\omega(\lambda)\omega(-\lambda) \quad &= \quad 1. \qquad (6.42)
\end{aligned}
$$

Therefore, for the last *layer* the eigenvalue is

$$
\Lambda_2(\lambda,\{\lambda_i^{(n-2)}\}|\{\lambda_i\}) = X_1(\lambda)\prod_{k=1}^{2} z(\lambda_k-\lambda) + X_3(\lambda)\prod_{k=1}^{2}\frac{x_2(\lambda-\lambda_k)}{y_{33}(\lambda-\lambda_k)} + X_2(\lambda)\prod_{k=1}^{2}\frac{z(\lambda-\lambda_k)}{\omega(\lambda-\lambda_k)} \qquad (6.43)
$$

provided that

$$
\frac{X_1(\lambda_a)}{X_2(\lambda_a)} = \frac{z(\lambda_a-\lambda_b)}{z(\lambda_b-\lambda_v)}\omega(\lambda_b-\lambda_a), \quad a \neq b = 1,2 \qquad (6.44)
$$

The factors X_a are given by

$$
X_3(\lambda) = \prod_{k=1}^{2} y_{33}(\lambda-\lambda_k^{n-2}) \qquad , X_a(\lambda) = \prod_{k=1}^{2} x_a(\lambda-\lambda_k^{n-2}), \qquad (a=1,2) \qquad (6.45)
$$

where the inhomogeneity parameter $\{\lambda_i^{n-2}\}$ make the link with the next to the last *layer*.

Let us go back to the *ground* in order to write the full nest through of a sequence of terms where the models are explicitly identified:

$$
\begin{aligned}
\Lambda_L(\lambda|\{\lambda_1,\lambda_2\}) \quad &= \quad X_1(\lambda)\prod_{k=1}^{2} z(\lambda_k-\lambda) + X_3(\lambda)\prod_{k=1}^{2}\frac{x_2(\lambda-\lambda_k)}{y_{NN}(\lambda-\lambda_k)} + X_2(\lambda)\frac{\Lambda_2^{(1)}(\lambda,\{\lambda_i\}|\{\lambda_i^{(1)}\})}{X_2^{(1)}(\lambda)} \\
(0) \quad &\in \quad (\mathrm{B}_n^{(1)},\mathrm{A}_{2n}^{(2)}) \qquad \text{and} \qquad (1) \in (\mathrm{B}_{n-1}^{(1)},\mathrm{A}_{2(n-1)}^{(2)})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Lambda_2^{(l)}(\lambda,\{\lambda_i^{(l-1)}\}|\{\lambda_i^{(l)}\})}{X_2^{(l)}(\lambda)} &= \frac{X_1^{(l)}(\lambda)}{X_2^{(l)}(\lambda)}\prod_{k=1}^{2} z^{(l)}(\lambda_k^{(l)}-\lambda) + \frac{X_3^{(l)}(\lambda)}{X_2^{(l)}(\lambda)}\prod_{k=1}^{2}\frac{x_2^{(l)}(\lambda-\lambda_k^{(l)})}{y_{N_l N_l}(\lambda-\lambda_k^{(l)})} \\
&\quad + \frac{\Lambda_2^{(l+1)}(\lambda,\{\lambda_i^{(l)}\}|\{\lambda_i^{(l+1)}\})}{X_2^{(l+1)}(\lambda)} \\
(l) &\in (\mathrm{B}_{n-l}^{(1)},\mathrm{A}_{2(n-l)}^{(2)}) \qquad \text{and} \qquad (l+1)\in(\mathrm{B}_{n-l-1}^{(1)},\mathrm{A}_{2(n-l-1)}^{(2)}) \\
l &= 1,2,...,n-2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Lambda_2^{(n-1)}(\lambda,\{\lambda_i^{(n-2)}\}|\{\lambda_i^{(n-1)}\})}{X_2^{(n-1)}(\lambda)} &= \frac{X_1^{(n-1)}(\lambda)}{X_2^{(n-1)}(\lambda)}\prod_{k=1}^{2} z^{(n-1)}(\lambda_k^{(n-1)}-\lambda) \\
&\quad + \frac{X_3^{(n-1)}(\lambda)}{X_2^{(n-1)}(\lambda)}\prod_{k=1}^{2}\frac{x_2^{(n-1)}(\lambda-\lambda_k^{(n-1)})}{y_{33}^{(n-1)}(\lambda-\lambda_k^{(n-1)})} \\
&\quad + \prod_{k=1}^{2}\frac{z^{(n-1)}(\lambda-\lambda_k^{(n-1)})}{\omega(\lambda-\lambda_k^{(n-1)})} \\
(n-1) &\in (\mathrm{B}_1^{(1)},\mathrm{A}_2^{(2)})
\end{aligned}
\tag{6.46}
$$

Remember that the inhomogeneity parameters $\{\lambda_i^{(l-1)}\}$ are implicit in (6.46) through of the definition of $X_a^{(l)}(\lambda)$, $a=1,2,3$. The function $\omega(\lambda-\lambda_k^{(n-1)})$ is given by (6.42).

The corresponding Bethe equations are

$$
\frac{X_1(\lambda_a)}{X_2(\lambda_a)} = \frac{z(\lambda_a-\lambda_b)}{z(\lambda_b-\lambda_a)}\frac{\Lambda_2^{(1)}(\lambda_a,\{\lambda_i\}|\{\lambda_i^{(1)}\})}{x_1^{(1)}(\lambda_a-\lambda_b)}, \quad a\neq b=1,2
$$

$$
\begin{aligned}
\frac{X_1^{(l)}(\lambda_a^{(l)})}{X_2^{(l)}(\lambda_a^{(l)})} &= \frac{z^{(l)}(\lambda_a^{(l)}-\lambda_b^{(l)})}{z^{(l)}(\lambda_b^{(l)}-\lambda_a^{(l)})}\frac{\Lambda_2^{(l+1)}(\lambda_a^{(l)},\{\lambda_i^{(l)}\}|\{\lambda_i^{(l+1)}\})}{x_1^{(l+1)}(\lambda_a^{(l)}-\lambda_b^{(l)})}, \quad a\neq b=1,2 \\
l &= 1,2,...,n-2
\end{aligned}
$$

$$
\frac{X_1^{(n-1)}(\lambda_a^{(n-1)})}{X_2^{(n-1)}(\lambda_a^{(n-1)})} = \frac{z^{(n-1)}(\lambda_a^{(n-1)}-\lambda_b^{(n-1)})}{z^{(n-1)}(\lambda_b^{(n-1)}-\lambda_a^{(n-1)})}\omega(\lambda_b^{(n-1)}-\lambda_a^{(n-1)}), \quad a\neq b=1,2 \tag{6.47}
$$

where

$$
\begin{aligned}
\Lambda_2^{(l+1)}(\lambda_a^{(l)},\{\lambda_i^{(l)}\}|\{\lambda_i^{(l+1)}\}) &= \operatorname*{res}_{\lambda=\lambda_a^{(l)}}\Lambda_2^{(l+1)}(\lambda,\{\lambda_i^{(l)}\}|\{\lambda_i^{(l+1)}\}) \\
&= x_1^{(l+1)}(0)x_1^{(l+1)}(\lambda_a^{(l)}-\lambda_b^{(l)})\prod_{k=1}^{2} z^{(l+1)}(\lambda_k^{(l+1)}-\lambda_a^{(l)}) \\
l &= 0,1,...,n-2, \quad a\neq b=1,2.
\end{aligned}
\tag{6.48}
$$

It is curious that the solution of the eigenvalue problem of the L site homogeneous transfer matrix for the two-particle state is given in terms of the eigenvalue problems of the two site inhomogeneous transfer matrices for two-particle state. We remark the participation of n different models in the construction of the nested Bethe ansatz for the $\mathrm{B}_n^{(1)}$ and $\mathrm{A}_{2n}^{(1)}$ vertex models.

6.2 $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{A}_{2n-1}^{(2)}$ two-particle state

For $N_l = 4, 6, ...$ the last *layer* involves the $\mathrm{C}_2^{(1)}$,$\mathrm{D}_2^{(1)}$ and $\mathrm{A}_3^{(2)}$ vertex models for which their nests are not complete. Indeed, just more one *layer* is necessary in order to complete them. To do this we recall (4.8) and (4.9) in order to include the $\mathrm{C}_1^{(1)}$,$\mathrm{D}_1^{(1)}$ and $\mathrm{A}_1^{(2)}$ vertex models in our discussion.

For the C_1 and $\mathrm{A}_1^{(2)}$ vertex models, the L site homogeneous transfer matrix is the trace of (4.9)

$$\tau_L(\lambda) = A_1(\lambda) + A_3(\lambda) \tag{6.49}$$

and the reference state

$$|0_L\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_L \tag{6.50}$$

is a highest vector of (4.9)

$$\begin{aligned} A_1(\lambda)\,|0_L\rangle &= X_1(\lambda)\,|0_L\rangle\,, \quad A_3(\lambda)\,|0_L\rangle = X_3(\lambda)\,|0_L\rangle \\ C_1(\lambda)\,|0_L\rangle &= 0, \quad B_1(\lambda)\,|0_L\rangle \neq \{0, |0_L\rangle\} \\ X_1(\lambda) &= [x_1(\lambda)]^L, \quad X_3(\lambda) = [y_{22}(\lambda)]^L \end{aligned} \tag{6.51}$$

The two-particle state is a new state without any relation with (6.4) which can be defined by

$$\Psi_2(\lambda_1, \lambda_2) = \Psi_2(\lambda_2, \lambda_1) = B_1(\lambda_1)B_1(\lambda_2) \tag{6.52}$$

The action of $\tau_L(\lambda)$ on this state can be computed using the commutation relations (6.5)-(6.7) in their reduced form

$$\begin{aligned} \tau_L(\lambda)\Psi_2(\lambda_1, \lambda_2) &= \left(X_1(\lambda) \prod_{k=1}^{2} \frac{x_1(\lambda_k - \lambda)}{y_{22}(\lambda_k - \lambda)} + X_3(\lambda) \prod_{k=1}^{2} \frac{x_1(\lambda - \lambda_k)}{y_{22}(\lambda - \lambda_k)} \right) \Psi_2(\lambda_1, \lambda_2) \\ &\quad - \frac{y_{12}(\lambda_1 - \lambda)}{y_{22}(\lambda_1 - \lambda)} \left(X_1(\lambda_1) \frac{x_1(\lambda_2 - \lambda_1)}{y_{22}(\lambda_2 - \lambda_1)} - X_3(\lambda_1) \frac{x_1(\lambda_1 - \lambda_2)}{y_{22}(\lambda_1 - \lambda_2)} \right) B_1(\lambda)B(\lambda_2)\,|0_L\rangle \\ &\quad - \frac{y_{12}(\lambda_2 - \lambda)}{y_{22}(\lambda_2 - \lambda)} \left(X_1(\lambda_2) \frac{x_1(\lambda_1 - \lambda_2)}{y_{22}(\lambda_1 - \lambda_2)} - X_3(\lambda_2) \frac{x_1(\lambda_2 - \lambda_1)}{y_{22}(\lambda_2 - \lambda_1)} \right) B_1(\lambda)B(\lambda_1)\,|0_L\rangle \end{aligned} \tag{6.53}$$

where we have used the property of the definition (6.52) and the identity

$$\frac{y_{12}(\lambda)}{y_{22}(\lambda)} + \frac{y_{21}(-\lambda)}{y_{22}(-\lambda)} = 0. \tag{6.54}$$

The eigenvalue is

$$\Lambda_L(\lambda|\lambda_1, \lambda_2) = X_1(\lambda) \prod_{k=1}^{2} \frac{x_1(\lambda_k - \lambda)}{y_{22}(\lambda_k - \lambda)} + X_3(\lambda) \prod_{k=1}^{2} \frac{x_1(\lambda - \lambda_k)}{y_{22}(\lambda - \lambda_k)} \tag{6.55}$$

provided that

$$\frac{X_1(\lambda_a)}{X_3(\lambda_a)} = \frac{x_1(\lambda_a - \lambda_b)}{x_1(\lambda_b - \lambda_a)} \frac{y_{22}(\lambda_b - \lambda_a)}{y_{22}(\lambda_a - \lambda_b)}, \quad a \neq b = 1, 2. \tag{6.56}$$

We now turn to the diagonalization problem of the $\mathrm{D}_2^{(1)}$ vertex model. It turns out, however, that the Lax operator of this model can decomposed in terms of Lax operator for the six-vertex model associated with the $\mathrm{A}_1^{(1)}$ Lie algebra. It means that from the isomorphism $\mathrm{D}_2^{(1)} = \mathrm{A}_1^{(1)} \oplus \mathrm{A}_1^{(1)}$, we can write

$$\mathcal{L}^{D_2}(\lambda) = \mathcal{L}_+^{A_1}(\lambda) \otimes \mathcal{L}_-^{A_1}(\lambda) \tag{6.57}$$

Here, a more careful analysis is required. First we identify the Lax operator $\mathcal{L}^{D_2}$ with the corresponding $\mathcal{R}$-matrix (2.1) and then we make a sign transformation in the Boltzmann weights $y_{\alpha\beta}(\lambda)$ that preserves the spectrum of the transfer matrix associated i.e., $y_{\alpha\beta}(\lambda) \to -y_{\alpha\beta}(\lambda)$ for $\alpha \neq \beta$ and $\alpha \neq \beta^{'}$. Now is not difficult to verify the isomorphism (6.57) where $\mathcal{L}^{A_1}(\lambda)$ is identified with the $\mathcal{R}$-matrix of the $\mathrm{A}_1^{(1)}$ vertex model listed in the our appendix.

Consequently, the eigenvalues of the model $\mathrm{D}_2^{(1)}$ are given in terms of the product of the eigenvalues of two $\mathrm{A}_1^{(1)}$ six-vertex models:

$$\Lambda_L^{D_2}(\lambda|\lambda_1^{\pm},\lambda_2^{\pm}) = \Lambda_L^{+}(\lambda|\lambda_1^{+},\lambda_2^{+})\Lambda_L^{-}(\lambda|\lambda_1^{-},\lambda_2^{-}) \tag{6.58}$$

where

$$\Lambda_L^{\pm}(\lambda|\lambda_1,\lambda_2) = X_1(\lambda)\prod_{k=1}^{2} z(\lambda_k^{\pm}-\lambda) + X_2(\lambda)\prod_{k=1}^{2} z(\lambda-\lambda_k^{\pm}) \tag{6.59}$$

provided that

$$\frac{X_1(\lambda_a^{\pm})}{X_2(\lambda_a^{\pm})} = \frac{z(\lambda_a^{\pm}-\lambda_b^{\pm})}{z(\lambda_b^{\pm}-\lambda_a^{\pm})}, \quad a \neq b = 1,2. \tag{6.60}$$

Here $\{\lambda_1^{+},\lambda_2^{+}\}$ and $\{\lambda_1^{-},\lambda_2^{-}\}$ are rapidities of the two-particle state related to each one of the two six-vertex models. The algebraic Bethe ansatz for $\mathrm{A}_n^{(1)}$ vertex models is discussed in the section 8 of this paper.

From these results we can see that the two particle state $\Psi_2(\lambda_1,\lambda_2)$ is an eigenstate of the homogeneous transfer matrix $\Lambda_L(\lambda)$ with eigenvalue

$$\begin{aligned}\Lambda_L(\lambda|\{\lambda_i\}) \quad &= \quad X_1(\lambda)\prod_{k=1}^{2} z(\lambda_k-\lambda) + X_3(\lambda)\prod_{k=1}^{2}\frac{x_2(\lambda-\lambda_k)}{y_{NN}(\lambda-\lambda_k)} \\ &\quad\; +X_2(\lambda)\left(\sum_{l=1}^{n-2} G_2^{(l)}(\lambda,\{\lambda_i^{(l-1)}\}|\{\lambda_i^{(l)}\}) + \mathcal{T}\right)\end{aligned} \tag{6.61}$$

where

$$G_2^{(l)}(\lambda,\{\lambda_i^{(l-1)}\}|\{\lambda_i^{(l)}\}) = \frac{X_1^{(l)}(\lambda)}{X_2^{(l)}(\lambda)}\prod_{k=1}^{2} z^{(l)}(\lambda_k^{(l)}-\lambda) + \frac{X_3^{(l)}(\lambda)}{X_2^{(l)}(\lambda)}\prod_{k=1}^{2}\frac{x_2^{(l)}(\lambda-\lambda_k^{(l)})}{y_{N_lN_l}^{(l)}(\lambda-\lambda_k^{(l)})} \tag{6.62}$$

and the last terms $\mathcal{T}$ depend on the model:

$$\mathcal{T} = G_2^{(n-1)}(\lambda,\{\lambda_i^{(n-2)}\}|\{\lambda_i^{(n-1)}\}) + \frac{X_1^{(n)}(\lambda)}{X_2^{(n)}(\lambda)}\prod_{k=1}^{2}\frac{x_1^{(n)}(\lambda_k^{(n)}-\lambda)}{y_{22}^{(n)}(\lambda_k^{(n)}-\lambda)} + \frac{X_3^{(n)}(\lambda)}{X_2^{(n)}(\lambda)}\prod_{k=1}^{2}\frac{x_1^{(n)}(\lambda-\lambda_k^{(n)})}{y_{22}^{(n)}(\lambda-\lambda_k^{(n)})} \tag{6.63}$$

for the $\mathrm{C}_n^{(1)}$ and $\mathrm{A}_{2n-1}^{(2)}$ vertex models and

$$\begin{aligned}\mathcal{T} \quad &= \quad [X_1^{(n-1)}(\lambda)\prod_{k=1}^{2} z^{(n-1)}(\lambda_k^{+}-\lambda) + X_2^{(n-1)}(\lambda)\prod_{k=1}^{2} z^{(n-1)}(\lambda-\lambda_k^{+})] \\ &\quad\; \times[X_1^{(n-1)}(\lambda)\prod_{k=1}^{2} z^{(n-1)}(\lambda_k^{-}-\lambda) + X_2^{(n-1)}(\lambda)\prod_{k=1}^{2} z^{(n-1)}(\lambda-\lambda_k^{-})] \\ (n-1) \quad &\in \quad \mathrm{A}_1^{(1)}\end{aligned} \tag{6.64}$$

for $D_n^{(1)}$ vertex model. The last row in (6.64) is to remember that the Boltzmann weights are those presented in the appendix.

The sequence of the Bethe equations has the form

$$\frac{X_1(\lambda_a)}{X_2(\lambda_a)}=\frac{z(\lambda_a-\lambda_b)}{z(\lambda_b-\lambda_a)}x_1^{(1)}(0)\prod_{k=1}^{2}z^{(1)}(\lambda_k^{(1)}-\lambda_a)\quad a\neq b=1,2$$

$$\begin{aligned}\frac{X_1^{(l)}(\lambda_a^{(l)})}{X_2^{(l)}(\lambda_a^{(l)})}&=\frac{z^{(l)}(\lambda_a^{(l)}-\lambda_b^{(l)})}{z^{(l)}(\lambda_b^{(l)}-\lambda_a^{(l)})}x_1^{(l+1)}(0)\prod_{k=1}^{2}z^{(l+1)}(\lambda_k^{(l+1)}-\lambda_a^{(l)}),\quad a\neq b=1,2\\ l&=1,2,...,n-2.\end{aligned}\tag{6.65}$$

which will end in two different ways:

$$\begin{aligned}\prod_{k=1}^{2}\frac{x_1^{(n-1)}(\lambda_a^{\pm}-\lambda_k^{(n-2)})}{x_2^{(n-1)}(\lambda_a^{\pm}-\lambda_k^{(n-2)})}&=\frac{z^{(n-1)}(\lambda_a^{\pm}-\lambda_b^{\pm})}{z^{(n-1)}(\lambda_b^{\pm}-\lambda_a^{\pm})},\quad a\neq b=1,2\\ (n-1)&\in \mathrm{A}_1^{(1)}\end{aligned}\tag{6.66}$$

for $D_n^{(1)}$ and more two equations

$$\begin{aligned}\frac{X_1^{(n-1)}(\lambda_a^{(n-1)})}{X_2^{(n-1)}(\lambda_a^{(n-1)})}&=\frac{z^{(n-1)}(\lambda_a^{(n-1)}-\lambda_b^{(n-1)})}{z^{(n-1)}(\lambda_b^{(n-1)}-\lambda_a^{(n-1)})}x_1^{(n)}(0)\prod_{k=1}^{2}z^{(n)}(\lambda_k^{(n)}-\lambda_a^{(n-1)}),\quad a\neq b=1,2\\ (n-1)&\in(\mathrm{C}_2^{(1)},\mathrm{A}_3^{(2)})\end{aligned}$$

$$\begin{aligned}\frac{X_1^{(n)}(\lambda_a^{(n)})}{X_3^{(n)}(\lambda_a^{(n)})}&=\frac{x_1^{(n)}(\lambda_a^{(n)}-\lambda_b^{(n)})}{x_1^{(n)}(\lambda_b^{(n)}-\lambda_a^{(n)})}\frac{y_{22}^{(n)}(\lambda_b^{(n)}-\lambda_a^{(n)})}{y_{22}^{(n)}(\lambda_a^{(n)}-\lambda_b^{(n)})},\quad a\neq b=1,2.\\ (n)&\in(\mathrm{C}_1^{(1)},\mathrm{A}_1^{(2)})\end{aligned}\tag{6.67}$$

for $C_n^{(1)}$ and $A_{2n-1}^{(2)}$ due to their continuation for $C_1^{(1)}$ and $A_1^{(2)}$ respectively. Here we are substituting the residue expressions for the eigenvalues given by (6.48).

The nested Bethe ansatz for one and two-particle state presented here contain all information necessary to known what happens when a multi-particle state is considered. The sequence of terms in the eigenvalues and in the Bethe equations are common for all models and differences will appear only in the last terms.

7 The multi-particle Bethe state

Generalization of previous results in order to consider Bethe states with more the two particles follows from [20] where the vector $\Phi_m^{(l)}$ was defined through the following recurrence formula:

$$\begin{aligned}&\Phi_m^{(l)}(\lambda_1,...,\lambda_m)=\mathcal{B}^{(l)}(\lambda_1)\otimes\Phi_{m-1}^{(l)}(\lambda_2,...,\lambda_m)\\ &-B_{N_l-1}(\lambda_1)\sum_{j=2}^{m}\frac{\hat{Y}_{N_l\ 2}^{(l)}(\lambda_1-\lambda_j)}{y_{N_lN_l}^{(l)}(\lambda_1-\lambda_j)}\otimes\Phi_{m-2}^{(l)}(\overset{\wedge}{\lambda_j})\prod_{k=2}^{j-1}\frac{S_{k,k+1}^{(l+1)}(\lambda_k-\lambda_j)}{x_1^{(l+1)}(\lambda_k-\lambda_j)}\prod_{\substack{k=2\\k\neq j}}^{m}z^{(l)}(\lambda_k-\lambda_j)A_1^{(l)}(\lambda_j)\end{aligned}\tag{7.1}$$

with the initial condition $\Phi_0^{(l)}=1,\quad \Phi_1^{(l)}(\lambda)=\mathcal{B}^{(l)}(\lambda)$. Here we have used the notation $(\overset{\wedge}{\lambda_j})$ to indicate the absence of the spectral parameter λ_j.

In order to proceed with the Bethe ansatz construction we must compute the action of the diagonal operators $A_i^{(l)}(\lambda), i=1,3$ and $\mathrm{Tr}_a[\mathcal{D}^{(l)}(\lambda)]$ of the monodromy matrix on the vectors $\Phi_m^{(l)}$. This procedure is really very laborious due to the normal-ordered condition but one can do it recursively. Acting with $A_1^{(l)}(\lambda)$ on (7.1) we have the following normal-ordered expression

$$
\begin{aligned}
&A_1^{(l)}(\lambda)\Phi_m^{(l)}(\lambda_1,...,\lambda_n)=\prod_{k=1}^{m} z^{(l)}(\lambda_k-\lambda)\Phi_m^{(l)}(\lambda_1,...,\lambda_m)A_1^{(l)}(\lambda) \\
&-\sum_{j=1}^{m}\frac{x_3^{(l)}(\lambda_j-\lambda)}{x_2^{(l)}(\lambda_j-\lambda)}\mathcal{B}^{(l)}(\lambda)\otimes\Phi_{m-1}^{(l)}(\overset{\wedge}{\lambda_j})\prod_{k=1}^{j-1}\frac{S_{k,k+1}^{(l+1)}(\lambda_k-\lambda_j)}{x_1^{(l+1)}(\lambda_k-\lambda_j)}\prod_{\substack{k=1\\k\neq j}}^{m} z^{(l)}(\lambda_k-\lambda_j)A_1^{(l)}(\lambda_j) \\
&+B_{N_l-1}(\lambda)\sum_{j=2}^{m}\sum_{p=1}^{j-1}\mathbf{G}_{jl}^{(l)}(\lambda,\lambda_p,\lambda_j)\otimes\Phi_{m-2}^{(l)}(\overset{\wedge}{\lambda_p},\overset{\wedge}{\lambda_j})\prod_{k=1}^{p-1}\frac{S_{k+1,k+2}^{(l+1)}(\lambda_k-\lambda_j)}{x_1^{(l+1)}(\lambda_k-\lambda_j)}\prod_{k=p+1}^{j-1}\frac{S_{k,k+1}^{(l+1)}(\lambda_k-\lambda_j)}{x_1^{(l+1)}(\lambda_k-\lambda_j)} \\
&\prod_{k=1}^{p-1}\frac{S_{k,k+1}^{(l+1)}(\lambda_k-\lambda_p)}{x_1^{(l+1)}(\lambda_k-\lambda_p)}\prod_{\substack{k=1\\k\neq p,j}}^{m} z^{(l)}(\lambda_k-\lambda_j)z^{(l)}(\lambda_k-\lambda_p)A_1^{(l)}(\lambda_p)A_1^{(l)}(\lambda_j)+\cdots
\end{aligned}
\tag{7.2}
$$

where the indices $k,k+1$ in the matrix $S_{k,k+1}^{(l+1)}$ denote the spaces where its action is not trivial. $\mathbf{G}_{jp}^{(l)}(\lambda,\lambda_l,\lambda_j)$ are matrix valued functions given by

$$
\mathbf{G}_{jp}^{(l)}(\lambda,\lambda_l,\lambda_j)=\frac{x_3^{(l)}(\lambda_j-\lambda)}{x_2^{(l)}(\lambda_j-\lambda)}\frac{\hat{Y}_{N_l\ 2}^{(l)}(\lambda-\lambda_p)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_p)}\frac{S^{(l+1)}(\lambda_p-\lambda)}{x_2^{(l+1)}(\lambda_p-\lambda)}+\frac{y_{1N_l}^{(l)}(\lambda_p-\lambda)}{y_{N_lN_l}^{(l)}(\lambda_p-\lambda)}\frac{\hat{Y}_{N_l\ 2}^{(l)}(\lambda_p-\lambda_j)}{y_{N_lN_l}^{(l)}(\lambda_p-\lambda_j)}
\tag{7.3}
$$

In (7.2) one can easily identify the candidate for the wanted term in the eigenvalue problem and two groups of unwanted terms. In each group terms differ by cyclic permutations of rapidities and consequently, by the presence of S matrices.

The action of $A_3^{(l)}(\lambda)$ on $\Phi_m^{(l)}$ has a similar form

$$
\begin{aligned}
&A_3^{(l)}(\lambda)\Phi_m^{(l)}(\lambda_1,...,\lambda_m)=\prod_{k=1}^{m}\frac{x_2^{(l)}(\lambda-\lambda_k)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_k)}\Phi_m^{(l)}(\lambda_1,...,\lambda_m)A_3^{(l)}(\lambda) \\
&-\sum_{j=1}^{m}\frac{\hat{Y}_{N_l\ 2}^{(l)}(\lambda-\lambda_j)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_j)}\mathcal{B}^{*(l)}(\lambda)\otimes\Phi_{m-1}^{(l)}(\overset{\wedge}{\lambda_j})\otimes\mathcal{D}^{(l)}(\lambda_j)\prod_{k=j+1}^{m}\frac{S_{k-1,k}^{(l+1)}(\lambda_j-\lambda_k)}{x_1^{(l+1)}(\lambda_j-\lambda_k)}\prod_{\substack{k=1\\k\neq j}}^{m} z^{(l)}(\lambda_j-\lambda_k) \\
&+B_{N_l-1}(\lambda)\sum_{j=2}^{m}\sum_{p=1}^{j-1}\mathbf{H}_{jp}^{(l)}(\lambda,\lambda_p,\lambda_j)\Phi_{m-2}^{(l)}(\overset{\wedge}{\lambda_p},\overset{\wedge}{\lambda_j})\otimes\mathcal{D}^{(l)}(\lambda_p)\otimes\mathcal{D}^{(l)}(\lambda_j)\prod_{k=j+1}^{m}\frac{S_{k-1,k}^{(l+1)}(\lambda_j-\lambda_k)}{x_1^{(l+1)}(\lambda_j-\lambda_k)} \\
\times\quad&\prod_{k=j+1}^{m}\frac{S_{k-2,k-1}^{(l+1)}(\lambda_l-\lambda_k)}{x_1^{(l+1)}(\lambda_l-\lambda_k)}\prod_{k=p+1}^{j-1}\frac{S_{k-1,k}^{(l+1)}(\lambda_p-\lambda_k)}{x_1^{(l+1)}(\lambda_l-\lambda_k)}\prod_{\substack{k=1\\k\neq p,j}}^{m} z^{(l)}(\lambda_p-\lambda_k)z^{(l)}(\lambda_j-\lambda_k)+\cdots
\end{aligned}
\tag{7.4}
$$

where we have the following matrix valued functions

$$
\mathbf{H}_{jp}^{(l)}(\lambda,\lambda_l,\lambda_j)=\frac{y_{N_l1}^{(l)}(\lambda-\lambda_p)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_p)}\frac{\hat{Y}_{N_l\ 2}^{(l)}(\lambda_p-\lambda_j)}{y_{N_l\ N_l}^{(l)}(\lambda_p-\lambda_j)}-\frac{x_3^{(l)}(\lambda-\lambda_p)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_p)}\frac{\hat{Y}_{N_l2}^{(l)}(\lambda-\lambda_j)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_j)}
\tag{7.5}
$$

In (7.4) we have a wanted term and presence of a new group of unwanted terms.

Acting with $\mathrm{Tr}_a[\mathcal{D}(\lambda)]$ on the vector $\Phi_m^{(l)}$ the final expression is more cumbersome

$$
\begin{aligned}
\mathrm{Tr}_a[\mathcal{D}^{(l)}(\lambda)]\Phi_m^{(l)}(\lambda_1,\dots,\lambda_m) &= \Phi_m^{(l)}(\lambda_1,...,\lambda_m)\mathrm{Tr}_a[\frac{\mathcal{L}_{am}^{(l+1)}(\lambda-\lambda_m)}{x_2^{(l+1)}(\lambda-\lambda_m)}\cdots\frac{\mathcal{L}_{a1}^{(l+1)}(\lambda-\lambda_1)}{x_2^{(l+1)}(\lambda-\lambda_1)}\mathcal{D}^{(l)}(\lambda)] \\
-\sum_{j=1}^{m}\frac{x_4^{(l)}(\lambda-\lambda_j)}{x_2^{(l)}(\lambda-\lambda_j)}B^{(l)}(\lambda)\otimes\Phi_{m-1}^{(l)}(\overset{\wedge}{\lambda_j})\mathcal{D}^{(l)}(\lambda_j)\otimes\mathbf{1}^{\otimes(m-1)}&\prod_{k=j+1}^{m}\frac{\mathcal{R}_{k-1,k}^{(l+1)}(\lambda_j-\lambda_k)}{x_1^{(l+1)}(\lambda_j-\lambda_k)}\prod_{\substack{k=1\\k\neq j}}^{m}z^{(l)}(\lambda_j-\lambda_k) \\
+\sum_{j=1}^{m}\frac{\hat{Y}_{N_l\;2}^{(l)}(\lambda-\lambda_j)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_j)}B^{*(l)}(\lambda)\otimes\Phi_{m-1}^{(l)}(\overset{\wedge}{\lambda_j})\otimes\mathbf{1}^{\otimes(m-2)}&\prod_{k=1}^{j-1}\frac{\mathcal{R}_{k,k+1}^{(l+1)}(\lambda_k-\lambda_j)}{x_1^{(l+1)}(\lambda_k-\lambda_j)}\prod_{\substack{k=1\\k\neq j}}^{m}z^{(l)}(\lambda_k-\lambda_j)A_1^{(l)}(\lambda_j) \\
+B_{N_l-1}(\lambda)\sum_{j=2}^{m}\sum_{p=1}^{j-1}\mathbf{Y}_{jp}^{(l)}(\lambda,\lambda_p,\lambda_j)\otimes\Phi_{m-2}^{(l)}(\overset{\wedge}{\lambda_p},\overset{\wedge}{\lambda_j})\mathcal{D}^{(l)}(\lambda_j)\otimes\mathbf{1}^{\otimes(m-1)}&\prod_{k=1}^{p-1}\frac{\mathcal{R}_{k,k+1}^{(l+1)}(\lambda_k-\lambda_p)}{x_1^{(l+1)}(\lambda_k-\lambda_p)} \\
\times\prod_{k=j+1}^{m}\frac{\mathcal{R}_{k-1,k}^{(l+1)}(\lambda_j-\lambda_k)}{x_1^{(l+1)}(\lambda_j-\lambda_k)}\prod_{\substack{k=1\\k\neq p.j}}^{m}z^{(l)}(\lambda_k-\lambda_p)z^{(l)}(\lambda_j-\lambda_k)A_1^{(l)}(\lambda_p)& \\
+B_{N_l-1}(\lambda)\sum_{j=2}^{m}\sum_{p=1}^{j-1}\mathbf{Y}_{jp}^{(l)}(\lambda,\lambda_j,\lambda_p)\otimes\Phi_{m-2}^{(l)}(\overset{\wedge}{\lambda_p},\overset{\wedge}{\lambda_j})\mathcal{D}^{(l)}(\lambda_p)\otimes\mathbf{1}^{\otimes(m-1)}&\prod_{k=1}^{l-1}\frac{\mathcal{R}^{(l+1)}(\lambda_k-\lambda_j)}{x_1^{(l+1)}(\lambda_k-\lambda_j)} \\
\times\prod_{k=j+1}^{m}\frac{\mathcal{R}_{k-1,k}^{(l+1)}(\lambda_p-\lambda_k)}{x_1^{(l+1)}(\lambda_p-\lambda_k)}\prod_{\substack{k=1\\k\neq p,j}}^{m}z^{(l)}(\lambda_k-\lambda_p)z^{(l)}(\lambda_j-\lambda_k)\frac{\mathcal{R}_{lj}^{(l+1)}(\lambda_p-\lambda_j)}{x_1^{(l+1)}(\lambda_p-\lambda_j)}A_1^{(l)}(\lambda_j)+\cdots& \qquad (7.6)
\end{aligned}
$$

where we have defined the matrix valued functions

$$
\mathbf{Y}_{jp}^{(l)}(\lambda,\lambda_p,\lambda_j)=[z^{(l)}(\lambda-\lambda_p)\frac{x_4^{(l)}(\lambda-\lambda_j)}{x_2^{(l)}(\lambda-\lambda_j)}-\frac{x_4^{(l)}(\lambda-\lambda_p)}{x_2^{(l)}(\lambda-\lambda_p)}\frac{x_4^{(l)}(\lambda_p-\lambda_j)}{x_2^{(l)}(\lambda_p-\lambda_j)}]\frac{\hat{Y}_{N_l2}^{(l)}(\lambda-\lambda_p)}{y_{N_lN_l}^{(l)}(\lambda-\lambda_p)} \qquad (7.7)
$$

Here we observe the presence of $\mathcal{R}$ matrices instead of S matrices as in the expressions for $A_1(\lambda)$ and $A_3(\lambda)$. This difference is fundamental for the Bethe ansatz construction. For instance, the trace (7.6) is to be understood as a $L+m$ site inhomogeneous transfer matrix with m inhomogeneous sites coming from the Lax operators $\mathcal{L}^{(l+1)}(\lambda-\lambda_k)\doteq\mathcal{R}^{(l+1)}(\lambda-\lambda_k)$. The solution of this inhomogeneous eigenvalue problem is the candidate for the wanted term in the eigenvalue problem of the L site homogeneous transfer matrix $\tau_L(\lambda)$. Moreover, the remained terms of (7.6) have the exact form to cancel the unwanted terms coming from (7.2) and (7.4). As before, the ellipses used in (7.2), (7.4) and (7.6) denote normally ordered terms containing annihilation operators.

In our nested Bethe ansatz language the *ground* ($l=0$) for a particular vertex model is prepared by a L site homogeneous transfer matrix $\tau_L(\lambda)$

$$
\tau_L(\lambda)=A_1(\lambda)+\sum_{\alpha=1}^{N-2}D_{\alpha\alpha}(\lambda)+A_3(\lambda) \qquad (7.8)
$$

and the multi-particle Bethe state is defined by the linear combination

$$
\Psi_m(\lambda_1,\cdots,\lambda_m)=\Phi_m(\lambda_1,...,\lambda_m)\mathcal{F}_m\left|0_L\right\rangle \qquad (7.9)
$$

where $\mathcal{F}_m$ is a vector matrix with $(N-2)^m$ entries $f^{\alpha_1\cdots\alpha_m}$.

The corresponding eigenvalue is obtained from the first term on the right hand side of (7.2), (7.4) and (7.6):

$$\Lambda_L(\lambda|\{\lambda_i\}) = X_1(\lambda)\prod_{k=1}^{m} z(\lambda_k-\lambda) + X_3(\lambda)\prod_{k=1}^{m}\frac{x_2(\lambda-\lambda_k)}{y_{NN}(\lambda-\lambda_k)} + X_2(\lambda)\frac{\Lambda_m^{(1)}(\lambda,\{\lambda_i\}|\{\lambda_i^{(1)}\})}{X_2^{(1)}(\lambda)} \tag{7.10}$$

Here we have used (3.9) and $\Lambda_m^{(1)}(\lambda,\{\lambda_i\}|\{\lambda_i^{(1)}\})$ is the eigenvalue of the eigenvalue problem for a m-site row-to-row inhomogeneous transfer matrix with its Lax operators identified with the matrices $\mathcal{R}^{(1)}(\lambda-\lambda_k)$, $k=1,...,m$ where λ_k are the inhomogenity parameters and $\{\lambda_i^{(1)}\}$ are rapidities:

$$\begin{aligned}\tau_m^{(1)}(\lambda)\mathcal{F}_m &= \mathrm{Tr}_a\left(\mathcal{L}_{am}^{(1)}(\lambda-\lambda_m)\cdots\mathcal{L}_{a1}^{(1)}(\lambda-\lambda_1)\right)\mathcal{F}_m \\ &= \Lambda_m^{(1)}(\lambda,\{\lambda_i\}|\{\lambda_i^{(1)}\})\mathcal{F}_m\end{aligned} \tag{7.11}$$

where the choice

$$\mathcal{F}_m = \Phi_m^{(1)}(\lambda_1^{(1)},...,\lambda_m^{(1)})\left|0_m^{(1)}\right\rangle \tag{7.12}$$

is implicit.

The remained terms of (7.2), (7.4) and (7.6) multiplied by $\mathcal{F}_m\left|0_L\right\rangle$ are known as unwanted terms. There are many of these terms but they can be collected in only three different groups. The first group contains m terms of the type $\mathcal{B}(\lambda)\otimes\Phi_{m-1}(\overset{\wedge}{\lambda_j})$. To see how is proceeding the cancel in this group, we start with $j=1$. The corresponding term has the form

$$-\frac{x_3(\lambda_1-\lambda)}{x_2(\lambda_1-\lambda)}\left(X_1(\lambda_1)\prod_{k=2}^{m} z(\lambda_k-\lambda_1) - X_2(\lambda_1)\prod_{k=2}^{m}\frac{\mathcal{R}_{k-1,k}^{(1)}(\lambda_1-\lambda_k)}{x_1^{(1)}(\lambda_1-\lambda_k)}\prod_{k=2}^{m} z(\lambda_1-\lambda_k)\right)\mathcal{F}_m\left|0_L\right\rangle \tag{7.13}$$

Now we take the limit of $\tau^{(1)}(\lambda)$ at $\lambda=\lambda_1$ in order to identify the product of $\mathcal{R}$ matrices

$$\begin{aligned}\tau_m^{(1)}(\lambda_1) &= \lim_{\lambda=\lambda_1}\tau_m^{(1)}(\lambda) = \lim_{\lambda=\lambda_1}\mathrm{Tr}_a\left(\mathcal{L}_{am}^{(1)}(\lambda-\lambda_m)\cdots\mathcal{L}_{a1}^{(1)}(\lambda-\lambda_1)\right) \\ &= \prod_{k=1}^{m-1}\mathcal{R}_{k,k+1}^{(1)}(\lambda_1-\lambda_k)\end{aligned} \tag{7.14}$$

Therefore we can use the solution of the second eigenvalue problem (7.11) at $\lambda=\lambda_1$ to see that this term will be cancelled provided that

$$\frac{X_1(\lambda_1)}{X_2(\lambda_1)}\mathcal{F}_m\left|0_L\right\rangle = \prod_{k=2}^{m}\frac{z(\lambda_1-\lambda_k)}{z(\lambda_k-\lambda_1)}\Lambda_m^{(1)}(\lambda_1,\{\lambda_i\}|\{\lambda_i^{(1)}\})\prod_{k=2}^{m}\frac{1}{x_1^{(1)}(\lambda_1-\lambda_k)}\mathcal{F}_m\left|0_L\right\rangle \tag{7.15}$$

The remained terms of this group can be written as cyclic permutations of (7.13). Thus, we can recall the relations (6.34)-(6.36) to prove that all terms of this group are eliminated provided that

$$\begin{aligned}\frac{X_1(\lambda_a)}{X_2(\lambda_a)} &= \left(\prod_{b\neq a}^{m}\frac{z(\lambda_a-\lambda_b)}{z(\lambda_b-\lambda_a)}\right)\frac{\Lambda_m^{(1)}(\lambda_a,\{\lambda_i\}|\{\lambda_i^{(1)}\})}{X^{(1)}(\lambda_a)} \\ a &= 1,2,...,m\end{aligned} \tag{7.16}$$

where

$$X^{(1)}(\lambda_a) = \prod_{b\neq a}^{m} x_1^{(1)}(\lambda_a-\lambda_b). \tag{7.17}$$

The second group contains m terms of the type $\mathcal{B}^*(\lambda) \otimes \Phi_{m-1}(\overset{\wedge}{\lambda}_j)$ and they are also cancelled by the *partial* Bethe equations (7.16). To accept this statement one can follow the same steps used in the first group. Finally, the third group contains of the type $B_{N-1}(\lambda) \otimes \Phi_{m-2}(\overset{\wedge}{\lambda}_p, \overset{\wedge}{\lambda}_j)$ are also cancelled by (7.16). Here we would like to stress that the technicalities involving the calculation of the third group of unwanted terms are very laborious.

At this point we have concluded the first step in an algebraic nested Bethe ansatz. The next step consists in taking into account the auxiliary eigenvalue problem (7.11). After we have presented the results for one and two particle states it is not frivolous to affirm that we will only need to repeat everything once more with trivial modifications. It is enough replace L by m and introduce the inhomogeneity parameters $\{\lambda_k^{(l-1)}\}$ and the rapidities $\{\lambda_k^{(l)}\}$ for each label l. Indeed this is true until we arrive at the last *layer* $l = n-1$, where the models behave differently.

The last *layer*s for the $\mathrm{B}_n^{(1)}$ and $\mathrm{A}_{2n}^{(2)}$ vertex models are building with the $\mathrm{B}_1^{(1)}$ Zamolodchivok-Fateev model and the $\mathrm{A}_2^{(2)}$ Izegin-Korepin model [23, 24], respectively. For $\mathrm{D}_n^{(1)}$ models the last *layer* is the $\mathrm{D}_2^{(1)}$ vertex model which is mapped in a direct product of two $\mathrm{A}_1^{(1)}$ six-vertex models. The $\mathrm{C}_n^{(2)}$ and $\mathrm{A}_{2n-1}^{(2)}$ models have their last *layer* extended to $l = n$ where we will find the $\mathrm{C}_1^{(1)}$ and $\mathrm{A}_1^{(2)}$ six-vertex models, respectively.

From these considerations we can summarize the results in the following way: the eigenvalue of the L site homogeneous transfer matrix $\tau_L(\lambda)$ for a m-particle Bethe state is given by

$$\begin{aligned}\Lambda_L(\lambda|\{\lambda_i\}) \quad = \quad & X_1(\lambda)\prod_{k=1}^{m} z(\lambda_k-\lambda) + X_3(\lambda)\prod_{k=1}^{m}\frac{x_2(\lambda-\lambda_k)}{y_{NN}(\lambda-\lambda_k)} \\ & +X_2(\lambda)\left(\sum_{l=1}^{n-2} G_m^{(l)}(\lambda,\{\lambda_i^{(l-1)}\}|\{\lambda_i^{(l)}\}) + \mathcal{T}\right)\end{aligned} \tag{7.18}$$

where

$$G_m^{(l)}(\lambda,\{\lambda_i^{(l-1)}\}|\{\lambda_i^{(l)}\}) = \frac{X_1^{(l)}(\lambda)}{X_2^{(l)}(\lambda)}\prod_{k=1}^{m} z^{(l)}(\lambda_k^{(l)}-\lambda) + \frac{X_3^{(l)}(\lambda)}{X_2^{(l)}(\lambda)}\prod_{k=1}^{m}\frac{x_2^{(l)}(\lambda-\lambda_k^{(l)})}{y_{N_lN_l}^{(l)}(\lambda-\lambda_k^{(l)})} \tag{7.19}$$

and we are again using the shorthand notation presented in (6.31) and (6.32). i.e.,

$$\begin{aligned} X_a(\lambda) &= [x_a(\lambda)]^L, \qquad X_a^{(l)}(\lambda) = \prod_{k=1}^{m} x_a^{(l)}(\lambda-\lambda_k^{(l-1)}) \qquad a=1,2. \\ X_3(\lambda) &= [y_{NN}(\lambda)]^L, \qquad X_3^{(l)}(\lambda) = \prod_{k=1}^{m} y_{N_lN_l}^{(l)}(\lambda-\lambda_k^{(l-1)})\end{aligned} \tag{7.20}$$

The $\mathcal{T}$ term in (7.18) makes the result difference for different models:

$$\mathcal{T} = G_m^{(n-1)}(\lambda,\{\lambda_i^{(n-2)}\}|\{\lambda_i^{(n-1)}\}) + \prod_{k=1}^{m}\frac{z^{(n-1)}(\lambda-\lambda_k^{(n-1)})}{\omega(\lambda-\lambda_k^{(n-1)})} \tag{7.21}$$

for $\mathrm{B}_n^{(1)}$ and $\mathrm{A}_{2n}^{(2)}$ vertex models,

$$\mathcal{T} = G_m^{(n-1)}(\lambda,\{\lambda_i^{(n-2)}\}|\{\lambda_i^{(n-1)}\}) + G_m^{(n)}(\lambda,\{\lambda_i^{(n-1)}\}|\{\lambda_i^{(n)}\}) \tag{7.22}$$

for $C_n^{(1)}$ and $A_{2n-1}^{(2)}$ vertex models. For $D_n^{(1)}$ vertex models the $\mathcal{T}$ term is a little bit different due to the direct product of two $A_1^{(1)}$ six-vertex models:

$$\begin{aligned}\mathcal{T} &= [X_1^+(\lambda)\prod_{k=1}^m z(\lambda_k^+ - \lambda) + X_2^+(\lambda)\prod_{k=1}^m z(\lambda - \lambda_k^+)] \\ &\quad \times [X_1^-(\lambda)\prod_{k=1}^m z(\lambda_k^- - \lambda) + X_2^-(\lambda)\prod_{k=1}^m z(\lambda - \lambda_k^-)]\end{aligned} \tag{7.23}$$

where

$$\begin{aligned} z(\lambda^{\pm}) &= \frac{x_1(\lambda^{\pm})}{x_2(\lambda^{\pm})}, \qquad X_a^{\pm}(\lambda) = \prod_{k=1}^m x_a(\lambda - \lambda_k^{n-3}), \qquad (a=1,2) \\ \{x_1, x_2\} &\in A_1^{(1)} \end{aligned} \tag{7.24}$$

The corresponding sequences of Bethe equations also have a common part

$$\frac{X_1(\lambda_a)}{X_2(\lambda_a)} = \prod_{b\neq a}^m \frac{z(\lambda_a - \lambda_b)}{z(\lambda_b - \lambda_a)} x_1^{(1)}(0)\prod_{k=1}^m z^{(1)}(\lambda_k^{(1)} - \lambda_a)$$

$$\begin{aligned}\frac{X_1^{(l)}(\lambda_a^{(l)})}{X_2^{(l)}(\lambda_a^{(l)})} &= \prod_{b\neq a}^m \frac{z^{(l)}(\lambda_a^{(l)} - \lambda_b^{(l)})}{z^{(l)}(\lambda_b^{(l)} - \lambda_a^{(l)})} x_1^{(l+1)}(0)\prod_{k=1}^m z^{(l+1)}(\lambda_k^{(l+1)} - \lambda_a^{(l)}), \\ l &= 1,2,...,n-2. \quad a\neq b = 1,2,...,m\end{aligned} \tag{7.25}$$

where we are substituting the residues

$$\begin{aligned}\Lambda_m^{(l+1)}(\lambda_a^{(l)}, \{\lambda_i^{(l)}\}|\{\lambda_i^{(l+1)}\}) &= \operatorname*{res}_{\lambda=\lambda_a^{(l)}} \Lambda_m^{(l+1)}(\lambda, \{\lambda_i^{(l)}\}|\{\lambda_i^{(l+1)}\}) \\ &= x_1^{(l+1)}(0) X_1^{(l+1)}(\lambda_a^{(l)})\prod_{k=1}^2 z^{(l+1)}(\lambda_k^{(l+1)} - \lambda_a^{(l)}) \\ l &= 0,1,...,n-2, \quad a\neq b = 1,2,...,m.\end{aligned} \tag{7.26}$$

As happened with the eigenvalue sequence (7.18) this sequence will end in different ways depending on the model:

$$\frac{X_1^{(n-1)}(\lambda_a^{(n-1)})}{X_2^{(n-1)}(\lambda_a^{(n-1)})} = \prod_{b\neq a}^m \frac{z^{(n-1)}(\lambda_a^{(n-1)} - \lambda_b^{(n-1)})}{z^{(n-1)}(\lambda_b^{(n-1)} - \lambda_a^{(n-1)})}\omega(\lambda_b^{(n-1)} - \lambda_a^{(n-1)}) \tag{7.27}$$

for $B_n^{(1)}$ and $A_{2n}^{(2)}$ vertex models,

$$\begin{aligned}\prod_{k=1}^m \frac{x_1(\lambda_a^{\pm} - \lambda_k^{(n-2)})}{x_2(\lambda_a^{\pm} - \lambda_k^{(n-2)})} &= \prod_{b\neq a}^m \frac{z(\lambda_a^{\pm} - \lambda_b^{\pm})}{z(\lambda_b^{\pm} - \lambda_a^{\pm})}, \\ \{x_1, x_2\} &\in A_1^{(1)}\end{aligned} \tag{7.28}$$

for $D_n^{(1)}$ vertex models. For $C_n^{(1)}$ and $A_{2n-1}^{(2)}$models we have more two equations: one equation for the *layer* $l = n-1$

$$\frac{X_1^{(n-1)}(\lambda_a^{(n-1)})}{X_2^{(n-1)}(\lambda_a^{(n-1)})} = \prod_{b\neq a}^m \frac{z^{(n-1)}(\lambda_a^{(n-1)} - \lambda_b^{(n-1)})}{z^{(n-1)}(\lambda_b^{(n-1)} - \lambda_a^{(n-1)})} x_1^{(n)}(0)\prod_{k=1}^m z^{(n)}(\lambda_k^{(n)} - \lambda_a^{(n-1)}),$$

and other equation for the *layer* $l = n$

$$\frac{X_1^{(n)}(\lambda_a^{(n)})}{X_3^{(n)}(\lambda_a^{(n)})} = \prod_{b \neq a}^{m} \frac{x_1^{(n)}(\lambda_a^{(n)} - \lambda_b^{(n)})}{x_1^{(n)}(\lambda_b^{(n)} - \lambda_a^{(n)})} \frac{y_{22}^{(n)}(\lambda_b^{(n)} - \lambda_a^{(n)})}{y_{22}^{(n)}(\lambda_a^{(n)} - \lambda_b^{(n)})} \tag{7.29}$$

Here we note that the function $\omega(\lambda)$ used in the $\mathrm{B}_n^{(1)}$ and $\mathrm{A}_{2n}^{(2)}$ was defined in (6.42).

There is a technical point which we already touched in the one-particle case. To solve the eigenvalue problem for a L site homogeneous transfer matrix $\tau_L(\lambda)$ with eigenstate $\Psi_m(\lambda_1, \lambda_2, \cdots, \lambda_m)$ we are left with a second eigenvalue problem for a m site inhomogeneous transfer matrix $\tau_m^{(1)}(\lambda, \{\lambda_i\})$ for which the vector $\mathcal{F}_m$ must be an eigenstate. However, $\mathcal{F}_m$ defines Ψ_m as a linear combinations of the components of the vector Φ_m. The dimensions of Φ_m , $|0_m\rangle$ and $\mathcal{F}_m$ are suggesting the choice of $\mathcal{F}_m$ as a m-particle state in the second eigenvalue problem. The choice of $\mathcal{F}_m$ as a r-particle state could give particular value for Ψ_m if $r < m$ and, increasing considerably the number of parameters $\{\lambda_i^{(l)}\}$ in each *layer* if $r > m$. However, these different choices are not necessary to fix the rapidities of the states via the Bethe equations.

8 Conclusion

In this paper the nested Bethe ansatz formulation is used to solve exactly a series of trigonometric vertex models based on the non-exceptional Lie algebras. Here a detailed account of this method was described in order to complement the results in the literature [17]-[20].

There are several issues for which this paper could useful:

a) The off-shell Bethe ansatz - Gaudin theory - Solution of the Kniznik-Zamolodchikov equation. The explicit expressions for the eigenvalue problem as presented in this paper define the off-shell Bethe ansatz equation. Now, if one can extend the Babujian and Flume formalism [26] for nested Bethe ansatz such a issue could be possible.

b) Graded matrix Bethe ansatz. A recent work [19] about the vertex models based on superalgebras assures the possibility to extend our result for graded vertex models.

c) A Bethe ansatz with open boundary conditions. The analytical Bethe ansatz for quantum-algebra-invariant spin chains [27, 28] gives us the eigenvalues for the models considered in this paper with quantum open boundary. The nested Bethe ansatz with diagonal reflection K-matrices was used by de Vega and González-Ruiz [29] to find eigenvectors and eigenvalues of the $\mathrm{A}_n^{(1)}$ vertex models. More recently, the nested Bethe ansatz with diagonal K-matrices boundary conditions for the $\mathrm{B}_n^{(1)}$ vertex model [30] and the $\mathrm{A}_{2n}^{(2)}$ vertex models [31] have been considered. Looking at the difficulties of obtaining these results we can see how could be useful an unified Bethe ansatz formulation with open boundary conditions.

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Appendix: $\mathbf{A}_n^{(1)}$ vertex models

For sake of completeness we will describe in this appendix how the $\mathrm{A}_n^{(1)}$ result is obtained by reduction from the our previous results

The $\mathrm{A}_n^{(1)}$ matrix $\mathcal{R}$ has the form

$$\begin{aligned} \mathcal{R}^{(l)} &= x_1 \sum_{\alpha \neq \alpha'}^{N_l} E_{\alpha\alpha} \otimes E_{\alpha\alpha} + x_2 \sum_{\alpha \neq \beta}^{N_l} E_{\alpha\alpha} \otimes E_{\beta\beta} + x_3 \sum_{\alpha < \beta}^{N_l} E_{\alpha\beta} \otimes E_{\beta\alpha} \\ &\quad + x_4 \sum_{\alpha > \beta}^{N_l} E_{\alpha\beta} \otimes E_{\beta\alpha} \end{aligned} \tag{A.1}$$

where the Boltzmann weights are given by

$$\begin{aligned} x_1(\lambda) &= \mathrm{e}^{\lambda} - q^2, \qquad x_2(\lambda) = q(\mathrm{e}^{\lambda} - 1) \\ x_3(\lambda) &= (q^2 - 1), \qquad x_4(\lambda) = \mathrm{e}^{\lambda} x_3(\lambda) \end{aligned} \tag{A.2}$$

Here $N_l = n - l + 1$. Note that these weights are not labeled by l because they are the same ones for all $\mathrm{A}_{n-l}^{(1)}$ models.

After we have identified the Lax operator with this $\mathcal{R}$ matrix the corresponding monodromy matrix can be written as a N_l by N_l matrix

$$T^{(l)} = \begin{pmatrix} A_1^{(l)} & B_1^{(l)} & B_2^{(l)} & \cdots & B_{N_l-1}^{(l)} \\ C_1^{(l)} & D_{11}^{(l)} & D_{12}^{(l)} & \cdots & D_{1,N_l-1}^{(l)} \\ C_2^{(l)} & D_{21}^{(l)} & D_{22}^{(l)} & \cdots & D_{2,N_l-1}^{(l)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{N_l-1}^{(l)} & D_{N_l-1,1}^{(l)} & D_{N_l-1,2}^{(l)} & \cdots & D_{N_l-1,N_l-1}^{(l)} \end{pmatrix} \tag{A.3}$$

The usual reference state in a L site homogeneous lattice is the highest vector of $T^{(l)}$

$$\begin{aligned} A_1^{(l)}(\lambda)\,|0_L\rangle^{(l)} &= X_1^{(l)}(\lambda)\,|0_L\rangle^{(l)}, \quad D_{\alpha\alpha}^{(l)}(\lambda)\,|0_L\rangle^{(l)} = X_2^{(l)}(\lambda)\,|0_L\rangle^{(l)} \\ C^{(l)}(\lambda)\,|0_L\rangle^{(l)} &= 0, \qquad\qquad\qquad D_{\alpha\beta}^{(l)}(\lambda)\,|0_L\rangle^{(l)} = 0 \\ B_{\alpha}^{(l)}(\lambda)\,|0_L\rangle^{(l)} &\neq \{0, |0_L\rangle^{(l)}\}, \qquad \alpha \neq \beta = 1, 2, ..., N_l - 1 \end{aligned} \tag{A.4}$$

where

$$X_1^{(l)}(\lambda) = [x_1(\lambda)]^L \qquad \text{and} \qquad X_2^{(l)}(\lambda) = [x_2(\lambda)]^L \tag{A.5}$$

Therefore we can write the monodromy matrix as a 2 by 2 matrix

$$T^{(l)}(\lambda) = \begin{pmatrix} A_1^{(l)}(\lambda) & \mathcal{B}^{(l)}(\lambda) \\ \mathcal{C}^{(l)}(\lambda) & \mathcal{D}^{(l)}(\lambda) \end{pmatrix} \tag{A.6}$$

where we identify a scalar $A^{(1)}(\lambda)$, two vector $\mathcal{B}^{(l)}(\lambda)$ and $\mathcal{C}^{(l)}(\lambda)$ with $N_l - 1$ entries and a $N_l - 1$ by $N_l - 1$ matrix $\mathcal{D}^{(l)}(\lambda)$.The commutation relations among the matrix elements of (A.6) can be obtained using the intertwining relation with

$$[S^{(l)}] = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_4 & x_2 & 0 \\ 0 & x_2 & x_3 & 0 \\ 0 & 0 & 0 & S^{(l+1)} \end{pmatrix} \tag{A.7}$$

where $S^{(l+1)}$ is exactly the permutation of (**??**) with l replaced by $l+1$. In (A.7) x_1 is scalar and $\{x_2, x_3, x_4\}$ are proportional to the identity matrix.

Now it is easy to see how our previous general results can be reduced in order to obtain the well-known results of the $\mathrm{A}_n^{(1)}$ vertex models [25]. Removing the third row and the third column of (3.11) we have (A.6). It means that the entries $\{B_{N_l-1}, \mathcal{B}^*, A_3, \mathcal{C}^*, C_{N_l-1}\}$ are vanishing in the $\mathrm{A}_n^{(1)}$ cases. Moreover, removing from (4.1) all row and column with entries $\{y_{\alpha\beta}\}$ we will get (A.7). Indeed these reductions already expected since that the structure of $\mathcal{R}$ matrices (2.1) and (A.1) differ (up to normalization) by the $y_{\alpha\beta}(\lambda)$ terms (2.3).

Using these reductions in the previous results we are working with the $\mathrm{A}_n^{(1)}$ vertex models. For instance, one can use the intertwining relation (4.5) with (A.6) and (A.7) in order to derive the following commutation relations

$$\begin{aligned}
A_1^{(l)}(\lambda)\mathcal{B}^{(l)}(\mu) &= z(\mu-\lambda)\mathcal{B}^{(l)}(\mu)A_1^{(l)}(\lambda) - \frac{x_3(\mu-\lambda)}{x_2(\mu-\lambda)}\mathcal{B}^{(l)}(\lambda)A_1^{(l)}(\mu) \\
\mathrm{Tr}_a[\mathcal{D}^{(l)}(\lambda)]B^{(l)}(\mu) &= \mathcal{B}^{(l)}(\mu)\mathrm{Tr}_a\left[\frac{\mathcal{L}^{(l+1)}(\lambda-\mu)}{x_2(\lambda-\mu)}\mathcal{D}^{(l)}(\lambda)\right] - \frac{x_4(\lambda-\mu)}{x_2(\lambda-\mu)}\mathcal{B}^{(l)}(\lambda)\mathcal{D}^{(l)}(\mu) \\
C^{(l)}(\lambda)\otimes B^{(l)}(\mu) &= \mathcal{B}^{(l)}(\mu)\otimes\mathcal{C}^{(l)}(\lambda) + \frac{x_4(\lambda-\mu)}{x_2(\lambda-\mu)}\left[A_1^{(l)}(\mu)\mathcal{D}^{(l)}(\lambda) - A_1^{(l)}(\lambda)\mathcal{D}^{(l)}(\mu)\right] \\
\mathcal{B}^{(l)}(\lambda)\otimes\mathcal{B}^{(l)}(\mu) &= \mathcal{B}^{(l)}(\mu)\otimes\mathcal{B}^{(l)}(\lambda)\frac{S^{(l+1)}(\lambda-\mu)}{x_1(\lambda-\mu)}
\end{aligned} \tag{A.8}$$

which are the reductions of (4.12), (5.2), (6.10) and (6.2), respectively

Now we define the normal ordered m-particle vector by the reduction of (7.1) to

$$\Phi_m^{(l)}(\lambda_1, \lambda_2, \dots, \lambda_m) = \mathcal{B}^{(l)}(\lambda_1)\otimes\mathcal{B}^{(l)}(\lambda_2)\otimes\cdots\otimes\mathcal{B}^{(l)}(\lambda_m) \tag{A.9}$$

Using (A.8) one can compute the action of the diagonal elements of (A.6) on this vector

$$\begin{aligned}
& A_1^{(l)}(\lambda)\Phi_m^{(l)}(\lambda_1, \lambda_2, \dots, \lambda_m) \\
= & \prod_{k=1}^{m} z(\lambda_k-\lambda)\Phi_m^{(l)}(\lambda_1, \lambda_2, \dots, \lambda_m)A_1^{(l)}(\lambda) \\
& -\sum_{j=1}^{m}\frac{x_3(\lambda_j-\lambda)}{x_2(\lambda_j-\lambda)}\mathcal{B}^{(l)}(\lambda)\otimes\Phi_{m-1}^{(l)}(\overset{\wedge}{\lambda_j})\prod_{k=1}^{j-1}\frac{S_{k,k+1}^{(l+1)}(\lambda_k-\lambda_j)}{x_1(\lambda_k-\lambda_j)}\prod_{\substack{k=1\\k\neq j}}^{m} z(\lambda_k-\lambda_j)A_1^{(l)}(\lambda_j)
\end{aligned} \tag{A.10}$$

and

$$\begin{aligned}
& \mathrm{Tr}_a[\mathcal{D}^{(l)}(\lambda)]\Phi_m^{(l)}(\lambda_1, \dots, \lambda_m) \\
= & \Phi_m^{(l)}(\lambda_1, ..., \lambda_m)\mathrm{Tr}_a[\frac{\mathcal{L}_{am}^{(l+1)}(\lambda-\lambda_m)}{x_2(\lambda-\lambda_m)}\cdots\frac{\mathcal{L}_{a1}^{(l+1)}(\lambda-\lambda_1)}{x_2(\lambda-\lambda_1)}\mathcal{D}^{(l)}(\lambda)] \\
& -\sum_{j=1}^{m}\frac{x_4(\lambda-\lambda_j)}{x_2(\lambda-\lambda_j)}B^{(l)}(\lambda)\otimes\Phi_{m-1}^{(l)}(\overset{\wedge}{\lambda_j})\mathcal{D}^{(l)}(\lambda_j)\otimes\mathbf{1}^{(l)\otimes(m-1)}\prod_{k=j+1}^{m}\frac{\mathcal{R}_{k-1,k}^{(l+1)}(\lambda_j-\lambda_k)}{x_1(\lambda_j-\lambda_k)}\prod_{\substack{k=1\\k\neq j}}^{m} z(\lambda_j-\lambda_k)
\end{aligned} \tag{A.11}$$

or we can recall (7.2) and (7.6) in order to get these expressions.

The eigenvalue problem on the *ground* ($l=0$) is

$$\begin{aligned}\tau_L(\lambda)\Psi(\lambda_1,\ldots,\lambda_m) &= \left(A(\lambda)+\mathrm{Tr}_a[\mathcal{D}(\lambda)]\right)\Psi(\lambda_1,\ldots,\lambda_m)\\ &= \Lambda_L(\lambda|\{\lambda_i\})\Psi(\lambda_1,\ldots,\lambda_m)\end{aligned} \tag{A.12}$$

where the m-particle Bethe state is defined by the linear combination

$$\Psi(\lambda_1,\ldots,\lambda_m)=\Phi_m(\lambda_1,\ldots,\lambda_m)\mathcal{F}_m\,|0_L\rangle\,.$$

Here, $\mathcal{F}_m$ is a vector with entries $f^{\alpha_1\cdots\alpha_m},\ \alpha_i=1,2,...,N-1$.

Substituting (A.10) and (A.11) into (A.12) the eigenvalue problem gets the form

$$\begin{aligned}\tau_L(\lambda)\Psi_m(\lambda_1,\ldots,\lambda_m) &= X_1(\lambda)\prod_{k=1}^{m}z(\lambda_k-\lambda)\Psi_m(\lambda_1,\lambda_2,\ldots,\lambda_m)\\ &\quad +X_2(\lambda)\frac{\Lambda_m^{(1)}(\lambda,\{\lambda_i\}|\{\lambda_i^{(1)}\})}{X_2^{(1)}(\lambda)}\Psi_m(\lambda_1,\lambda_2,\ldots,\lambda_m)\\ &\quad -\sum_{j=1}^{m}\frac{x_3(\lambda_j-\lambda)}{x_2(\lambda_j-\lambda)}\mathcal{B}(\lambda)\otimes\Phi_{m-1}(\overset{\wedge}{\lambda_j})\\ &\quad \times[M_m^{(1)}(\lambda_j,\{\lambda_i\})]^{j-1}\left\{X_1(\lambda_j)\prod_{k=1,k\neq j}^{m}z(\lambda_k-\lambda_j)\right.\\ &\quad \left.-X_2(\lambda_j)\prod_{k=1,k\neq j}^{m}z(\lambda_j-\lambda_k)\frac{\Lambda_m^{(1)}(\lambda_j,\{\lambda_i\}|\{\lambda_i^{(1)}\})}{X_1^{(1)}(\lambda_j)}\right\}\mathcal{F}_m\,|0_L\rangle\end{aligned} \tag{A.13}$$

where we have made the choice

$$\mathcal{F}_m=\Phi_m^{(1)}(\lambda_1^{(1)},\ldots,\lambda_m^{(1)})\,|0_m\rangle^{(1)} \tag{A.14}$$

and we are collected the unwanted terms taking into account the cyclic permutation property (6.36).

Therefore, the eigenvalue is

$$\Lambda_L(\lambda|\{\lambda_i\})=X_1(\lambda)\prod_{k=1}^{m}z(\lambda_k-\lambda)+X_2(\lambda)\frac{\Lambda_m^{(1)}(\lambda,\{\lambda_i\}|\{\lambda_i^{(1)}\})}{X_2^{(1)}(\lambda)} \tag{A.15}$$

provided

$$\begin{aligned}\frac{X_1(\lambda_j)}{X_2(\lambda_j)} &= \left(\prod_{k=1,k\neq j}^{m}\frac{z(\lambda_j-\lambda_k)}{z(\lambda_k-\lambda_j)}\right)\frac{\Lambda_m^{(1)}(\lambda_j,\{\lambda_i\}|\{\lambda_i^{(1)}\})}{X_1^{(1)}(\lambda_j)}\\ &= \prod_{k=1,k\neq j}^{m}\frac{z(\lambda_j-\lambda_k)}{z(\lambda_k-\lambda_j)}x_1(0)\prod_{k\neq j}^{m}z(\lambda_k^{(1)}-\lambda_j)\end{aligned} \tag{A.16}$$

where we have substitute $\Lambda_m^{(1)}(\lambda_j,\{\lambda_i\}|\{\lambda_i^{(1)}\})$ by the residue of $\Lambda_m^{(1)}(\lambda,\{\lambda_i\}|\{\lambda_i^{(1)}\})$ at $\lambda=\lambda_j$ which is find by the second eigenvalue problem. The second eigenvalue problem gives us

$$\frac{\Lambda_m^{(1)}(\lambda,\{\lambda_i\}|\{\lambda_i^{(1)}\})}{X_2^{(1)}(\lambda)}=\frac{X_1^{(1)}(\lambda)}{X_2^{(1)}(\lambda)}\prod_{k=1}^{m}z(\lambda_k^{(1)}-\lambda)+\frac{\Lambda_m^{(2)}(\lambda,\{\lambda_i^{(1)}\}|\{\lambda_i^{(2)}\})}{X_2^{(2)}(\lambda)} \tag{A.17}$$

provided that

$$
\begin{aligned}
\frac{X_1^{(1)}(\lambda_j^{(1)})}{X_2^{(1)}(\lambda_j^{(1)})} &= \left(\prod_{k=1,k\neq j}^{m} \frac{z(\lambda_j^{(1)}-\lambda_k^{(1)})}{z(\lambda_k^{(1)}-\lambda_j^{(1)})}\right) \frac{\Lambda_m^{(2)}(\lambda_j^{(1)},\{\lambda_i^{(1)}\}|\{\lambda_i^{(2)}\})}{X_1^{(2)}(\lambda_j^{(1)})} \\
&= \prod_{k=1,k\neq j}^{m} \frac{z(\lambda_j^{(1)}-\lambda_k^{(1)})}{z(\lambda_k^{(1)}-\lambda_j^{(1)})} x_1(0) \prod_{k\neq j}^{m} z(\lambda_k^{(2)}-\lambda_j^{(1)})
\end{aligned}
$$

where we have substitute $\Lambda_m^{(2)}(\lambda_j^{(1)},\{\lambda_i^{(1)}\}|\{\lambda_i^{(2)}\})$ by the residue of $\Lambda_m^{(2)}(\lambda,\{\lambda_i^{(1)}\}|\{\lambda_i^{(2)}\})$ at $\lambda=\lambda_j^{(1)}$ which is given by the third eigenvalue problem, and so on. We follow this procedure till we reach the last layer which consists of the six-vertex model whose transfer-matrix diagonalization is well known in the literature. Therefore, the eigenvalue of the transfer matrix for the $\mathrm{A}_n^{(1)}$ vertex models is given by

$$
\Lambda_L(\lambda|\{\lambda_i\}) = X_1(\lambda)\prod_{k=1}^{m} z(\lambda_k-\lambda) + X_2(\lambda)\left(\sum_{l=1}^{n-1} G_m^{(l)}(\lambda,\{\lambda_i^{(l-1)}\}|\{\lambda_i^{(l)}\}) + \prod_{k=1}^{m} z(\lambda-\lambda_k^{(n-1)})\right) \tag{A.18}
$$

where

$$
G_m^{(l)}(\lambda,\{\lambda_i^{(l-1)}\}|\{\lambda_i^{(l)}\}) = \frac{X_1^{(l)}(\lambda)}{X_2^{(l)}(\lambda)}\prod_{k=1}^{m} z(\lambda_k^{(l)}-\lambda) \tag{A.19}
$$

The Bethe equations are

$$
\begin{aligned}
\frac{X_1(\lambda_j)}{X_2(\lambda_j)} &= = \prod_{k=1,k\neq j}^{m} \frac{z(\lambda_j-\lambda_k)}{z(\lambda_k-\lambda_j)} x_1(0) \prod_{k\neq j}^{m} z(\lambda_k^{(1)}-\lambda_j) \\
\frac{X_1^{(l)}(\lambda_j^{(l)})}{X_2^{(l)}(\lambda_j^{(l)})} &= \prod_{k=1,k\neq j}^{m} \frac{z(\lambda_j^{(l)}-\lambda_k^{(l)})}{z(\lambda_k^{(l)}-\lambda_j^{(l)})} x_1(0) \prod_{k\neq j}^{m} z(\lambda_k^{(l+1)}-\lambda_j^{(l)}) \\
l &= 1,2,...,n-1
\end{aligned} \tag{A.20}
$$

These results complete our study about the nested Bethe ansatz for vertex models based in non-exceptional Lie algebras.

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